International Journal of Industrial Engineering & Production Research

March 2011, Volume 22 Number 1
pp. 51-62

http://IJIEPR.iust.ac.ir/

ISSN: 2008-4889

"Technical Note"

A Fuzzy TOPSIS Method Based on Left and Right Scores

Mir. B. Aryanezhad, M.J. Tarokh, M.N. Mokhtarian* & F. Zaheri

Mir. B. Aryanezhad, *Department of Industrial Engineering, Iran University of Science and Technology, Tehran, Iran*
M.J. Tarokh, *Department of Industrial Engineering, K . N. Toosi University of Technology, Tehran, Iran*
** **M.N. Mokhtarian,*** *Department of Industrial Engineering, Najafabad Branch, Islamic Azad University, Isfahan, Iran*
F. Zaheri, *Department of Industrial Engineering, Sanandaj Branch, Islamic Azad University, Sanandaj, Iran*

KEYWORDS

MCDM; TOPSIS;
Fuzzy sets;
Left and right scores

ABSTRACT

Multiple criteria decision making (MCDM) problem is one of the famous different kinds of decision making problems. In more cases in real situations, determining the exact values for MCDM problems is difficult or impossible. So, the values of alternatives with respect to the criteria or / and the values of criteria weights, are considered as fuzzy values (fuzzy numbers). In such conditions, the conventional crisp approaches for solving MCDM problems tend to be less effective for dealing with the imprecise or vagueness nature of the linguistic assessments. In this situation, the fuzzy MCDM methods are applied for solving MCDM problems. In this paper, we propose a fuzzy TOPSIS (for Order Preference by Similarity to Ideal Solution) method based on left and right scores for fuzzy MCDM problems. To show the applicability of the proposed method, two numerical examples are presented. As a result, our proposed method is precise, easy use and practical for solving MCDM problem with fuzzy data. Moreover, the proposed method considers the decision makers (DMs) preference in the decision making process. It seems that the proposed fuzzy TOPSIS method is flexible and easy to use and has a low computational volume.

1. Introduction

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method is a popular approach to multiple criteria decision making (MCDM) problems that was proposed by Hwang and Yoon [1]. This method has been widely used in the literature.
Some of papers in the literature applied the TOPSIS method for solving real application problems. For example: Wang and Elhag [2] proposed a fuzzy TOPSIS method based on alpha level sets and presented a nonlinear programming (NLP) solution procedure. The relationship between the fuzzy TOPSIS method and fuzzy weighted average (FWA) is also discussed in their paper. Kahraman et al. [3] developed a multi-attribute decision making model for evaluating and selecting among logistic information technologies. They also presented a sensitivity analysis for their method. Boran et al. [4] combined TOPSIS method with intuitionistic fuzzy set to select appropriate supplier in group decision making environment. Intuitionistic fuzzy weighted averaging (IFWA) operator utilized to aggregate individual opinions of decision makers for rating the importance of criteria and alternatives. Amiri et al [5] presented a new method which is extracted from the multiple decision making methods named eigenvector–DEA–TOPSIS method to evaluate the risk of the number of related

* **Corresponding author. M.N. Mokhtarian**
Email: *mn_mokhtarian@yahoo.com mn_mokhtarian@iaun.ac.ir*
Paper first received May. 15. 2010, and in revised form Jan. 10. 2011.

portfolios to the foreign exchange market (FOREX). Chamodrakas et al. [6] proposed a new class of fuzzy methods for evaluating customers is. Firstly, their approach tackles the issue of uncertainty that is inherent in the problem of customer evaluation and secondly, the TOPSIS method is modified in order to integrate the behavioral pattern of the decision maker into its ''principle of compromise". Sun and Lin [7] used the fuzzy TOPSIS method based on fuzzy sets in solving MCDM problems. From their research results, the security and trust are the most important factors for improving the competitive advantage of shopping website. Gumus [8] proposed a two step methodology to evaluate hazardous waste transportation firms containing the methods of fuzzy-AHP and TOPSIS. In their method, TOPSIS uses fuzzy-AHP result weights as its input weights. Lin et al. [9] presented a framework that integrates the AHP and the TOPSIS to assist designers in identifying customer requirements and design characteristics, and help to achieve an effective evaluation of the final design solution. Onut et al. [10] presented a fuzzy TOPSIS based methodology to solve the solid waste transshipment site selection problem in Istanbul, Turkey. In their method, the criteria weights are calculated by using the AHP.

Some of papers in the literature applied the TOPSIS method in theoric context. For example: Shih [11] proposed 11-step procedure to exploit incremental analysis or marginal analysis to overcome the drawbacks of ratio scales utilized invarious multi-criteria or multi-attribute decision making (MCDM/MADM) techniques. Deng et al. [12] formulated the inter-company comparison process as a multi-criteria analysis model, and presented an effective approach by modifying TOPSIS for solving the problem. The modified TOPSIS approach can identify the relevance of the financial ratios to the evaluation result, and indicate the performance difference between companies on each financial ratio. Chen [13] extended a fuzzy TOPSIS method so that the rating of each alternative and the weight of each criterion are described by linguistic terms which can be expressed in triangular fuzzy numbers. Then, a vertex method is proposed to calculate the distance between two triangular fuzzy numbers. Opriovic and Tzeng [14] compared two MCDM methods, VIKOR and TOPSIS, focusing on modeling aggregating function and normalization, in order to reveal and to compare the procedural basis of these two MCDM methods. Wang and Lee [15] proposed a novel approach that involves end-user into the whole decision making process. In their approach, the subjective weights assigned by decision makers (DM) are normalized into a comparable scale. In addition, they also adopted end-user ratings as an objective weight based on Shannon's entropy theory. Ashtiani et al. [16] proposed interval-valued fuzzy TOPSIS method for solving MCDM problems in which the weights of criteria are unequal, using interval-valued fuzzy sets concepts. Jahanshahloo et al. [17] presented a new method for solving MCDM problems by TOPSIS method consisting of interval data. In their method the score of each alternative will be an interval number. Triantaphyllou and Lin [18] developed a fuzzy version of the TOPSIS method based on fuzzy arithmetic operations, which leads to a fuzzy relative closeness for each alternative.

The rest of this paper is organized as follows. In section 2, we only introduced the backgrounds that significantly related to this paper. Additional backgrounds are given in the appendixes A and B .In section 3 , we propose our fuzzy TOPSIS method based on left and right scores. In section 4, two numerical examples are presented to better explain the capabilities and potentials of our proposed method. Finally, the paper is concluded in section 5.

2. Background

We propose the normalization process for triangular and trapezoidal fuzzy numbers, separately as follow.

If $\tilde{y}_{ij} = (a_{ij}, b_{ij}, c_{ij})$ $(i = 1,\ldots,n,\ j = 1,\ldots,m)$ are triangular fuzzy numbers, then the normalization process can be conducted as:

$$\left(\tilde{y}_{ij}\right)_N = \left(\left(a_{ij}\right)_N, \left(b_{ij}\right)_N, \left(c_{ij}\right)_N\right) = \left(\frac{a_{ij} - a_j^{Min}}{\Delta_{Min}^{Max}}, \frac{b_{ij} - a_j^{Min}}{\Delta_{Min}^{Max}}, \frac{c_{ij} - a_j^{Min}}{\Delta_{Min}^{Max}}\right), \quad i = 1,\ldots,n;\ j \in \Omega_b \tag{1}$$

$$\left(\tilde{y}_{ij}\right)_N = \left(\left(a_{ij}\right)_N, \left(b_{ij}\right)_N, \left(c_{ij}\right)_N\right) = \left(\frac{c_{ij} - c_j^{Max}}{\Delta_{Max}^{Min}}, \frac{b_{ij} - c_j^{Max}}{\Delta_{Max}^{Min}}, \frac{a_{ij} - c_j^{Max}}{\Delta_{Max}^{Min}}\right), \quad i = 1,\ldots,n;\ j \in \Omega_c \tag{2}$$

Where Ω_b and Ω_c are the sets of benefit criteria / attributes and cost criteria / attributes, respectively and

$$c_j^{Max} = Max c_{ij} \quad , \quad a_j^{Min} = Min a_{ij} \quad , \quad i = 1,\ldots,n$$
$$\Delta_{Min}^{Max} = c_j^{Max} - a_j^{Min} \quad , \quad \Delta_{Max}^{Min} = a_j^{Min} - c_j^{Max}$$

If $\tilde{y}_{ij}=(a_{ij},b_{ij},c_{ij},d_{ij})\ \ (i=1,\ldots,n,\ j=1,\ldots,m)$ are trapezoidal fuzzy numbers, then the normalization process can be conducted as:

$$(\tilde{y}_{ij})_N=((a_{ij})_N,(b_{ij})_N,(c_{ij})_N,(d_{ij})_N)=\left(\frac{a_{ij}-a_j^{Min}}{\Delta_{Min}^{Max}},\frac{b_{ij}-a_j^{Min}}{\Delta_{Min}^{Max}},\frac{c_{ij}-a_j^{Min}}{\Delta_{Min}^{Max}},\frac{d_{ij}-a_j^{Min}}{\Delta_{Min}^{Max}}\right)\quad i=1,\ldots,n\ ;\ j\in\Omega_b \tag{3}$$

$$(\tilde{y}_{ij})_N=((a_{ij})_N,(b_{ij})_N,(c_{ij})_N,(d_{ij})_N)=\left(\frac{d_{ij}-d_j^{Max}}{\Delta_{Max}^{Min}},\frac{c_{ij}-d_j^{Max}}{\Delta_{Max}^{Min}},\frac{b_{ij}-d_j^{Max}}{\Delta_{Max}^{Min}},\frac{a_{ij}-d_j^{Max}}{\Delta_{Max}^{Min}}\right)\quad i=1,\ldots,n\ ;\ j\in\Omega_c \tag{4}$$

Where Ω_b and Ω_c are the sets of benefit criteria and cost criteria, respectively and

$$d_j^{Max}=Max d_{ij}\ ,\quad a_j^{Min}=Min a_{ij}\ ,\quad i=1,\ldots,n$$
$$\Delta_{Min}^{Max}=d_j^{Max}-a_j^{Min}\ ,\quad \Delta_{Max}^{Min}=a_j^{Min}-d_j^{Max}$$

It should be consider that in above equations, $(\tilde{y}_{ij})_N$ $(i=1,\ldots,n,j=1,\ldots,m)$ are the normalized fuzzy numbers related to fuzzy numbers $\tilde{y}_{ij}\ \ (i=1,\ldots,n,\ j=1,\ldots,m)$.

Due to the fact that our proposed method in this paper is completely related to left and right scores of fuzzy numbers, in the following, we express basic concepts about left and right scores of fuzzy numbers. The left and right scores were first introduced by Chen [20] for ranking fuzzy numbers.

When we have fuzzy number $\tilde{K}$, the left and right scores refer to the intersection of a fuzzy number $\tilde{K}$ with the fuzzy min and the fuzzy max , respectively. Fig. 1. illustrates the mentioned notion (using fuzzy max and fuzzy min that defined by Chen and Hwang [20]) graphically.

Where $(L_S)_{\tilde{K}}$ and $(R_S)_{\tilde{K}}$ are left and right scores, respectively. Generally, the left and right scores for fuzzy number $\tilde{K}$ are defined as

$$(L_S)_{\tilde{K}}=\sup_x\left[\mu_{\tilde{K}}(x)\wedge\mu_{\min}(x)\right] \tag{5}$$

$$(R_S)_{\tilde{K}}=\sup_x\left[\mu_{\tilde{K}}(x)\wedge\mu_{\max}(x)\right] \tag{6}$$

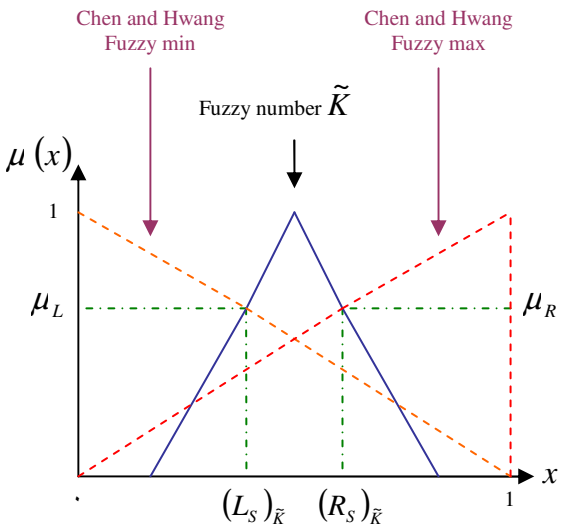

Fig. 1. The left and right scores for fuzzy number $\tilde{K}$.

And membership functions for maximizing set and minimizing set are defined as:

$$\mu_{\max}(x)=\begin{cases}x & ,\ 0\le x\le 1\\ 0 & ,\ otherwise\end{cases}\quad,\quad \mu_{\min}(x)=\begin{cases}1-x & ,\ 0\le x\le 1\\ 0 & ,\ otherwise\end{cases} \tag{7}$$

In order to simplification and for determining left and right scores for normalized fuzzy numbers $(\tilde{y}_{ij})_N$ $(i=1,\ldots,n,\ j=1,\ldots,m)$, with respect to the Eqs. (5) and (6), we define Eqs. (8) and (9) or Eqs. (10) and (11) as

$$(L_S)_{ij}=\frac{(b_{ij})_N}{1+(b_{ij})_N-(a_{ij})_N} \tag{8}$$

$$(R_S)_{ij}=\frac{(c_{ij})_N}{1+(c_{ij})_N-(b_{ij})_N} \tag{9}$$

$$(L_S)_{ij} = \frac{(b_{ij})_N}{1+(b_{ij})_N - (a_{ij})_N} \quad (10)$$

$$(R_S)_{ij} = \frac{(d_{ij})_N}{1+(d_{ij})_N - (c_{ij})_N} \quad (11)$$

According to Shipley et al. [22], in the general TOPSIS method, the separation of each alternative from the ideal solution (D_i^*) and the separation of each alternative from the negative ideal solution (D_i^-) are interrelated with the criteria weights and the criteria weights should be incorporated in the distances measurements. This is because all alternatives are compared with I^* and I^-, rather than directly among themselves. So, in order to eliminate this drawback, Deng et al. [23] presented the weighted Euclidean distance instead of creating a weighted decision matrix. Due to these facts, modified ideal (I^*) and negative ideal (I^-) solutions are defined as:

$$I^* = \{((y_1)_N)^*,...,((y_m)_N)^*\} = \left\{(\underset{j}{Max}\ (y_{ij})_N \mid j\in\Omega_b), (\underset{j}{Min}\ (y_{ij})_N \mid j\in\Omega_c)\right\}$$
$$I^- = \{((y_1)_N)^-,...,((y_m)_N)^-\} = \left\{(\underset{j}{Min}\ (y_{ij})_N \mid j\in\Omega_b), (\underset{j}{Max}\ (y_{ij})_N \mid j\in\Omega_c)\right\} \quad (12)$$

Where Ω_b and Ω_c are defined as before.

Moreover, modified separations of each alternative from the ideal solution (D_i^*) and negative ideal solution (D_i^-) are defined as

$$D_i^* = \sqrt{\sum_{j=1}^{m} w_j \left((y_{ij})_N - ((y_j)_N)^*\right)^2} \quad (13)$$

$$D_i^- = \sqrt{\sum_{j=1}^{m} w_j \left((y_{ij})_N - ((y_j)_N)^-\right)^2} \quad (14)$$

3. Fuzzy TOPSIS Method Based on Left and Right Scores

Suppose a crisp MCDM problem has *n* alternatives $(A_1,\ldots,A_n)$ and *m* decision criteria $(C_1,\ldots,C_m)$. Each alternative is evaluated with respect to the *m* criteria. All the values / ratings assigned to the alternatives with respect to each criterion from a decision matrix, denoted by $S=(y_{ij})_{n\times m}$, and the relative weight vector about the criteria, denoted by $W=(w_1,\ldots,w_m)$, that satisfying $\sum_{j=1}^{m} w_j = 1$. Due to the fact that, in some cases, determining the exact values for the elements of decision matrix is difficult, so, their values are considered as fuzzy numbers. In the other words, in fuzzy MCDM problems, the values of alternatives with respect to each criterion and the values of relative weights with respect to each criterion are usually characterized by fuzzy numbers. By considering the fact that, the TOPSIS method can also be used to deal with MCDM problems as a popular, accurate, and easy to use method, in this section, we extend TOPSIS for fuzzy MCDM problems based on left and right scores via an algorithmic method as follows:

Step 1: Construct the fuzzy decision matrix and fuzzy weights matrix as:

$$\tilde{Y} = \begin{pmatrix} \tilde{y}_{11} & \cdots & \tilde{y}_{1j} & \cdots & \tilde{y}_{1m} \\ \vdots & & \vdots & & \vdots \\ \tilde{y}_{i1} & \cdots & \tilde{y}_{ij} & \cdots & \tilde{y}_{im} \\ \vdots & & \vdots & & \vdots \\ \tilde{y}_{n1} & \cdots & \tilde{y}_{nj} & \cdots & \tilde{y}_{nm} \end{pmatrix}, \quad \tilde{W} = (\tilde{w}_1, \ \ldots \ , \tilde{w}_j, \ \ldots \ , \tilde{w}_m) \quad (15)$$

Step 2: Normalize fuzzy decision matrix by Eqs. (1) and (2) or Eqs. (3) and (4) and normalize fuzzy weights matrix by Eqs. (1) or (3). The normalized fuzzy decision matrix and normalized fuzzy weights matrix are shown as

$$(\tilde{y})_N = \begin{pmatrix} (\tilde{y}_{11})_N & \cdots & (\tilde{y}_{1j})_N & \cdots & (\tilde{y}_{1m})_N \\ \vdots & & \vdots & & \vdots \\ (\tilde{y}_{i1})_N & \cdots & (\tilde{y}_{ij})_N & \cdots & (\tilde{y}_{im})_N \\ \vdots & & \vdots & & \vdots \\ (\tilde{y}_{n1})_N & \cdots & (\tilde{y}_{nj})_N & \cdots & (\tilde{y}_{nm})_N \end{pmatrix} \quad (16)$$

$$(\tilde{W})_N = \left((\tilde{w}_1)_N, \ \ldots \ , (\tilde{w}_j)_N, \ \ldots \ , (\tilde{w}_m)_N\right)$$

In above equations, we define $(\tilde{y}_{ij})_N$ and $(\tilde{w}_j)_N$ as normalized fuzzy values / ratings related to $\tilde{y}_{ij}$ and $\tilde{w}_j$ respectively. Moreover, one of the important issues in our proposed method is that, the fuzzy weights matrix should be consider as a separate column and should be normalize by Eqs. (1) or (3).

Step 3: Calculate the left and right scores of normalized fuzzy numbers by Eqs. (8) and (9) or Eqs. (10) and (11). It should be consider that the normalized fuzzy numbers are existent in normalized fuzzy decision matrix and normalized fuzzy weights matrix.

Step 4: Construct two matrices that include intervals of left and right scores. It should be noted that one of the matrices is related to normalized fuzzy decision matrix, another is related to normalized fuzzy weights matrix. The Two matrices are shown as:

$$((L_S),(R_S))_{(\tilde{r})_N} = \begin{pmatrix} [(L_S),(R_S)]_{11} & \cdots & [(L_S),(R_S)]_{1j} & \cdots & [(L_S),(R_S)]_{1m} \\ \vdots & & \vdots & & \vdots \\ [(L_S),(R_S)]_{i1} & \cdots & [(L_S),(R_S)]_{ij} & \cdots & [(L_S),(R_S)]_{im} \\ \vdots & & \vdots & & \vdots \\ [(L_S),(R_S)]_{n1} & \cdots & [(L_S),(R_S)]_{nj} & \cdots & [(L_S),(R_S)]_{nm} \end{pmatrix} \quad (17)$$

$$((L_S),(R_S))_{(\tilde{W})_N} = \left([(L_S),(R_S)]_1, \ \ldots \ , [(L_S),(R_S)]_j, \ \ldots \ , [(L_S),(R_S)]_m\right) \quad (18)$$

For the purpose of better realization, we show the concept of $[(L_S),(R_S)]$ graphically by Fig. 2.

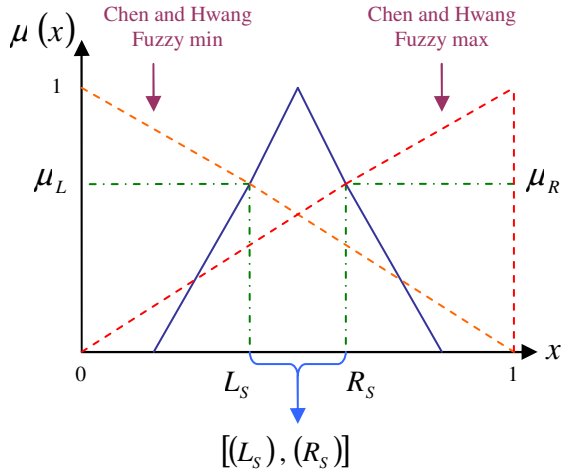

Fig. 2. The concept of $[(L_S),(R_S)]$.

Step 5: Determine the matrices that include ideal and negative ideal solutions as

$$I^* = (\tilde{1},\ldots,\tilde{1}) \quad (19)$$

$$I^- = (\tilde{0},\ldots,\tilde{0}) \quad (20)$$

Where $\tilde{1} = (1,1,1)$ and $\tilde{0} = (0,0,0)$.
It is obvious that the normalized criteria values / ratings are between zero and one. So, we consider the matrix (I^*), that all members are one, as ideal solutions matrix, and the matrix (I^-), that all members are zero, as negative ideal solution matrix.

Step 6: Calculate the separations of each alternative from the ideal solution (D_i^*) and negative ideal solution (D_i^-).

If we consider $r_{ij} = [(L_S),(R_S)]_{ij} = \lfloor (L_S)_{ij}, (R_S)_{ij} \rfloor$ and $w_j = [(L_S),(R_S)]_j = \lfloor (L_S)_j, (R_S)_j \rfloor$, and with respect to the Eqs. (19), we can write the separations of each alternative from the ideal solution (D_i^*) and negative ideal solution (D_i^-) as

$$D_i^* = \sqrt{\sum_{j=1}^{m} w_j (r_{ij} - 1)^2} \quad (21)$$

$$D_i^- = \sqrt{\sum_{j=1}^{m} w_j (r_{ij} - 0)^2} = \sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2} \quad (22)$$

Where

$$(L_S)_j \le w_j \le (R_S)_j \quad , \quad j = 1, \dots, m$$

$$(L_S)_{ij} \le r_{ij} \le (R_S)_{ij} \quad , \quad j = 1, \dots, m$$

$$(RC)_i = \frac{\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j (r_{ij}-1)^2}} \quad , \quad i = 1, \dots, n \tag{23}$$

$$(L_S)_j \le w_j \le (R_S)_j \quad , \quad j = 1, \dots, m$$

$$(L_S)_{ij} \le r_{ij} \le (R_S)_{ij} \quad , \quad j = 1, \dots, m$$

It is clear that, $(RC)_i$ is an interval that whose lower and upper bounds can be captured by the multi objective fractional decision making model as:

$$((RC)_i)^U = Max \frac{\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j (r_{ij}-1)^2}} \tag{24}$$

$$\frac{\partial (RC)_i}{\partial (r_{ij})} = \frac{w_j (r_{ij}) \sqrt{\frac{\sum_{j=1}^{m} w_j (r_{ij}-1)^2}{\sum_{j=1}^{m} w_j (r_{ij})^2}} + w_j (1-r_{ij}) \sqrt{\frac{\sum_{j=1}^{m} w_j (r_{ij})^2}{\sum_{j=1}^{m} w_j (r_{ij}-1)^2}}}{\left(\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j (r_{ij}-1)^2} \right)^2} > 0 \quad , \quad \text{j} = 1, \dots, \text{m} \tag{25}$$

It is clear that $(RC)_i$ is a monotonically increasing function of r_{ij}, that reaches its maximum at $r_{ij} = (R_S)_{ij}$ and its minimum at $r_{ij} = (L_S)_{ij}$. So, the above multi objective fractional decision making model can be rewritten as:

$$((RC)_i)^U = Max \frac{\sqrt{\sum_{j=1}^{m} w_j ((R_S)_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j ((R_S)_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j ((R_S)_{ij}-1)^2}} \tag{26}$$

$$(RC)_i = Max \left(\frac{\alpha \times \sqrt{\sum_{j=1}^{m} w_j ((R_S)_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j ((R_S)_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j ((R_S)_{ij}-1)^2}} - \frac{\beta \times \sqrt{\sum_{j=1}^{m} w_j ((L_S)_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j ((L_S)_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j ((L_S)_{ij}-1)^2}} \right) \tag{27}$$

s.t.

$$(L_S)_j \le w_j \le (R_S)_j \quad , \quad j = 1, \dots, m$$

Step 7: Calculate the relative closeness interval of each alternative to the ideal solution.

With respect to the Eqs. (21) and (22), we can write the relative closeness of each alternative to the ideal solution as:

$$((RC)_i)^L = Min \frac{\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j (r_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j (r_{ij}-1)^2}}$$

s.t.

$$(L_S)_j \le w_j \le (R_S)_j \quad , \quad j = 1, \dots, m$$

$$(L_S)_{ij} \le r_{ij} \le (R_S)_{ij} \quad , \quad j = 1, \dots, m$$

Due to the fact that:

$$((RC)_i)^L = Min \frac{\sqrt{\sum_{j=1}^{m} w_j ((L_S)_{ij})^2}}{\sqrt{\sum_{j=1}^{m} w_j ((L_S)_{ij})^2} + \sqrt{\sum_{j=1}^{m} w_j ((L_S)_{ij}-1)^2}}$$

s.t.

$$(L_S)_j \le w_j \le (R_S)_j \quad , \quad j = 1, \dots, m$$

So, in the decision making process, if the upper bound and lower bound objectives have importance (weights) equal to α and β respectively, the above multi objective fractional decision making model can be converted to:

So, it is obvious that the $(RC)_i$ for each alternative can be generated by solving the NLP model (26). Moreover the above NLP model can be solved using Microsoft Excel solver or LINGO software package without striking a blowing because its constraints are all linear.
It should be considered that, we apply the $\alpha = \beta = 0.5$ for solving the numerical examples in this paper.

Step 8: Rank and prioritize the alternatives according to their relative closeness to the ideal solution.
As a summary, the new fuzzy TOPSIS method based on left and right scores can be summed up as follows:

- Construct the fuzzy decision matrix and fuzzy weights matrix.
- Normalize fuzzy decision matrix by Eqs. (1) and (2) or Eqs. (3) and (4) and normalize fuzzy weights matrix by Eqs. (1) or (3).
- Calculate the left and right scores of fuzzy numbers by Eqs. (8) and (9) or Eqs. (10) and (11). It should be consider that fuzzy numbers are existent in normalized fuzzy decision matrix and normalized fuzzy weights matrix.
- Construct two matrices that include intervals of left and right scores. It should be noted that one of the matrices is related to normalized fuzzy decision matrix, another is related to normalized fuzzy weights matrix.
- Determine the matrices that include ideal and negative ideal solutions.
- Calculate the separations of each alternative from the ideal solution (D_i^*) and negative ideal solution (D_i^-).
- Calculate the relative closeness of each alternative to the ideal solution.
- Rank and prioritize the alternatives according to their relative closeness to the ideal solution.

4. Numerical Examples

In this section, we examine two numerical examples using our method for the purpose of showing the applicability of it. The first numerical example is taken from Triantaphyllou and Lin [25]. The second numerical example is a real application related to the agent selection in the supply chain of an electronic company.

Example 1. Reconsider the example investigated by Triantaphyllou and Lin [18], in which three alternatives A_1~A_3 are evaluated against four benefit criteria C_1~C_4. Table 1 shows the fuzzy weights, normalized fuzzy weights and left and right scores related to them. Table 2 shows fuzzy ratings of alternatives, normalized fuzzy ratings of alternatives and left and right scores related to them. The relative closeness of each alternative to the ideal solution and the average relative closeness of each alternative to the ideal solution and the rank of each alternative are shown in Table 3. We also compare the given results of our method with the results of Wang et al's [2] and Triantaphyllou and Lin's [18] methods for this numerical example. We also propose the comparing results in the Table 3.

Tab. 1. Fuzzy weights, normalized fuzzy weights and left and right scores

Criteria	Fuzzy weights	Normalized fuzzy weights	Left and right scores
C1	(0.13, 0.20, 0.31)	(0.10, 0.25, 0.48)	[0.22, 0.39]
C2	(0.08, 0.15, 0.25)	(0.00, 0.15, 0.35)	[0.13, 0.29]
C3	(0.29, 0.40, 0.56)	(0.44, 0.67, 1.00)	[0.54, 0.75]
C4	(0.17, 0.25, 0.38)	(0.19, 0.35, 0.63)	[0.30, 0.49]

Tab. 2. Fuzzy ratings of alternatives, normalized fuzzy ratings and left and right scores

Criteria	Alternatives	Fuzzy ratings of alternatives	Normalized fuzzy ratings of alternatives	Left and right scores
C1	A1	(0.08, 0.25, 0.94)	(0.00, 0.06, 0.28)	[0.05, 0.23]
	A2	(0.23, 1.00, 3.10)	(0.05, 0.30, 1.00)	[0.24, 0.59]
	A3	(0.15, 0.40, 1.48)	(0.02, 0.11, 0.46)	[0.10, 0.34]
C2	A1	(0.25, 0.93, 2.96)	(0.04, 0.28, 1.00)	[0.23, 0.58]
	A2	(0.13, 0.60, 2.24)	(0.00, 0.17, 0.75)	[0.14, 0.47]
	A3	(0.13, 0.20, 0.88)	(0.00, 0.02, 0.27)	[0.02, 0.21]
C3	A1	(0.34, 0.70, 1.71)	(0.09, 0.20, 0.50)	[0.18, 0.38]
	A2	(0.03, 0.05, 0.09)	(0.00, 0.01, 0.02)	[0.01, 0.02]
	A3	(0.62, 1.48, 3.41)	(0.17, 0.43, 1.00)	[0.34, 0.64]
C4	A1	(0.12, 0.24, 0.92)	(0.00, 0.04, 0.27)	[0.04, 0.22]
	A2	(0.12, 0.40, 1.48)	(0.00, 0.10, 0.47)	[0.09, 0.34]
	A3	(0.24, 1.00, 3.03)	(0.04, 0.30, 1.00)	[0.24, 0.59]

Tab. 3. The relative closeness of each alternative to the ideal solution & the average relative closeness of each alternative to the ideal solution and the rank of each alternative

Alternatives	The proposed method		Wang et al's method		Triantaphyllou and Lin's method	
	Relative closeness to the ideal solution	Rank and prioritize	Relative closeness to the ideal solution	Rank and prioritize	Relative closeness to the ideal solution	Rank and prioritize
A1	0.1106	2	0.2380	2	(0.04, 0.42, 5.83)	2
A2	0.1088	3	0.2021	3	(0.01, 0.21, 3.99)	3
A3	0.1394	1	0.4207	1	(0.06, 0.79, 10.42)	1

As shown in Table 3, the relative closenesses for three alternatives are 0.1106 for A_1, 0.1088 for A_2 and 0.1394 for A_3. Which lead to the ranking of $A_3 \succ A_1 \succ A_2$. Where the symbol '$\succ$' means 'is superior or preferred to'.

With respect to the taken results, our method leads to the same ranking as Wang et al's Triantaphyllou and Lin's methods. It should be considered that the taken results in our method are significantly related to the decision maker (DM) preferences.

Example 2. Recently, following the economic crisis in the world, Economic conditions in many countries and especially in developing countries has been very critical. In this situation the large companies are forced to keep their survival via reducing their cost. Due to this fact, one of the best ways to reduce the cost for these companies is to reduce the production volume and the number of their agents in their supply chain. It should be considered that, the large companies often spend a lot of cost for their agents.

In this section, we propose a real multiple criteria decision making problem in which a famous electronic company in Iran wants to reduce agents from 6 agents to 2 in the Isfahan city. It should be considered that each agent is located at the especial zone different from the others. For this purpose, the electronic company uses the consulting team that includes three consultants (three decision makers (DMs)).

The consultants consider 6 agents and evaluate them against eight criteria which include: credibility of agent with respect to the retailers (C_1), average time of ordering for each agent (in terms of week) (C_2), easy shipment of product to each agent (C_3), working time duration for each agent (in terms of month) (C_4), the number of agents of competing company in each zone (C_5), the density of the electronic parts retailers in each zone (C_6), the location position for each agent that include: being in sight (C_7), possibility of future development (C_8). It should be consider that the ratings of each alternative with respect to the criteria C_2, C_4 and C_5 are expressed by crisp values while the ratings of alternatives with respect to other criteria are expressed by linguistic variables that are defined in Table 5. On the other hand, except criterion $C_2$and C_5, all of the other criteria are benefit criteria. The relative importance weights of the nine criteria are described by linguistic variables which are defined in Table 4. Table 6 and 7show the original assessment information provided by three consultants (three DMs), where aggregated fuzzy numbers are obtained by averaging the fuzzy opinions of three consultants that is $\tilde{w}_j = \frac{\left(\tilde{w}_j^1 + \tilde{w}_j^2 + \tilde{w}_j^3\right)}{3}$ and $\tilde{x}_{ij} = \frac{\left(\tilde{x}_{ij}^1 + \tilde{x}_{ij}^2 + \tilde{x}_{ij}^3\right)}{3}$, where $\tilde{w}_j^k$ and $\tilde{x}_{ij}^k$ are the relative importance weight and the values given by the kth consultant (kth DM). Moreover, the left and right scores of aggregated fuzzy weights are shown in Table 6 and the aggregated ratings of each alternative with respect to each criterion and the left and right scores of them are shown in Table 7. The relative closeness of each alternative to the ideal solution and the rank of each alternative are shown in Table 8.

Tab. 4. Linguistic variables for the relative importance weights of criteria

Linguistic variable	Fuzzy number
Very low (VL)	(0.00, 0.10, 0.20)
Low (L)	(0.10, 0.20, 0.30)
Medium low (ML)	(0.20, 0.40, 0.50)
Medium (M)	(0.40, 0.50, 0.60)
Medium high (MH)	(0.50, 0.70, 0.80)
High (H)	(0.70, 0.80, 0.90)
Very high (VH)	(0.80, 1.00, 1.00)

Tab. 5. Linguistic variables for the ratings of alternatives

Linguistic variable	Fuzzy number
Very poor (VP)	(0.00, 1.00, 2.00)
poor (P)	(1.00, 2.00, 3.00)
Medium poor (MP)	(2.00, 4.00, 5.00)
Fair (F)	(4.00, 5.00, 6.00)
Medium good (MG)	(5.00, 7.00, 8.00)
Good (G)	(7.00, 8.00, 9.00)
Very good (VG)	(8.00, 10.00, 10.00)

Tab. 6. The relative importance weights of nine criteria, aggregated fuzzy weights and the left and right scores of aggregated fuzzy weights

Criteria	DM 1	DM 2	DM 3	Aggregated fuzzy weights	Left and right scores
C_1	VH	H	VH	(0.77, 0.93, 0.97)	[0.80, 0.93]
C_2	H	MH	H	(0.63, 0.77, 0.87)	[0.68, 0.79]
C_3	MH	H	H	(0.63, 0.77, 0.87)	[0.68, 0.79]
C_4	MH	MH	MH	(0.50, 0.70, 0.80)	[0.58, 0.73]
C_5	H	H	H	(0.70, 0.80, 0.90)	[0.73, 0.82]
C_6	VH	VH	H	(0.77, 0.93, 0.97)	[0.80, 0.93]
C_7	H	H	MH	(0.63, 0.77, 0.87)	[0.68, 0.79]
C_8	H	H	VH	(0.73, 0.87, 0.93)	[0.76, 0.88]

Tab. 7. The ratings of alternatives with respect to the nine criteria, aggregated fuzzy ratings and the left and right scores of aggregated fuzzy ratings

Criteria	Alternatives	DMs			Aggregated fuzzy number	Left and right scores
		DM 1	DM 2	DM 3		
C_1	A_1	VG	G	VG	(7.67, 9.33, 9.67)	[0.71, 0.94]
	A_2	MG	G	VG	(6.67, 8.33, 9.00)	[0.57, 0.78]
	A_3	F	MG	MG	(4.67, 6.33, 7.33)	[0.29, 0.47]
	A_4	F	F	MG	(4.33, 5.67, 6.67)	[0.20, 0.37]
	A_5	G	VG	VG	(7.67, 9.33, 9.67)	[0.71, 0.94]
	A_6	MG	F	MG	(4.67, 6.33, 7.33)	[0.29, 0.47]
C_2	A_1	6.00	6.00	6.00	(6.00, 6.00, 6.00)	[0.20, 0.20]
	A_2	5.00	5.00	5.00	(5.00, 5.00, 5.00)	[0.40, 0.40]
	A_3	5.00	5.00	5.00	(5.00, 5.00, 5.00)	[0.40, 0.40]
	A_4	7.00	7.00	7.00	(7.00, 7.00, 7.00)	[0.00, 0.00]
	A_5	2.00	2.00	2.00	(2.00, 2.00, 2.00)	[1.00, 1.00]
	A_6	4.00	4.00	4.00	(4.00, 4.00, 4.00)	[0.60, 0.60]
C_3	A_1	MG	F	F	(4.33, 5.67, 6.67)	[0.51, 0.63]
	A_2	VG	G	VG	(7.67, 9.33, 9.67)	[0.82, 0.97]
	A_3	P	VP	VP	(0.33, 1.33, 2.33)	[0.12, 0.22]
	A_4	MP	F	MP	(2.67, 4.33, 5.33)	[0.38, 0.50]
	A_5	VP	VP	VP	(0.00, 1.00, 2.00)	[0.09, 0.19]
	A_6	G	G	VG	(7.33, 8.67, 9.33)	[0.79, 0.90]
C_4	A_1	10.00	10.00	10.00	(10.00, 10.00, 10.00)	[0.00, 0.00]
	A_2	27.00	27.00	27.00	(27.00, 27.00, 27.00)	[0.27, 0.27]
	A_3	60.00	60.00	60.00	(60.00, 60.00, 60.00)	[0.79, 0.79]
	A_4	73.00	73.00	73.00	(73.00, 73.00, 73.00)	[1.00, 1.00]
	A_5	60.00	60.00	60.00	(60.00, 60.00, 60.00)	[0.79, 0.79]
	A_6	20.00	20.00	20.00	(20.00, 20.00, 20.00)	[0.16, 0.16]
C_5	A_1	1.00	1.00	1.00	(1.00, 1.00, 1.00)	[1.00, 1.00]
	A_2	3.00	3.00	3.00	(3.00, 3.00, 3.00)	[0.71, 0.71]
	A_3	5.00	5.00	5.00	(5.00, 5.00, 5.00)	[0.43, 0.43]
	A_4	2.00	2.00	2.00	(2.00, 2.00, 2.00)	[0.86, 0.86]
	A_5	8.00	8.00	8.00	(8.00, 8.00, 8.00)	[0.00, 0.00]
	A_6	2.00	2.00	2.00	(2.00, 2.00, 2.00)	[0.86, 0.86]
	A_1	MP	MP	MP	(2.00, 4.00, 5.00)	[0.21, 0.35]
	A_2	MG	G	G	(6.33, 7.67, 8.67)	[0.63, 0.77]

C_6	A_3	G	MG	G	(6.33, 7.67, 8.67)	[0.63, 0.77]
	A_4	MG	F	F	(4.33, 5.67, 6.67)	[0.41, 0.54]
	A_5	VG	VG	G	(7.67, 9.33, 9.67)	[0.79, 0.96]
	A_6	F	MG	F	(4.33, 5.67, 6.67)	[0.41, 0.54]
C_7	A_1	VG	G	G	(7.33, 8.67, 9.33)	[0.70, 0.89]
	A_2	MG	G	MG	(5.67, 7.33, 8.33)	[0.48, 0.68]
	A_3	G	MG	G	(6.33, 7.67, 8.67)	[0.55, 0.74]
	A_4	MG	F	F	(4.33, 5.67, 6.67)	[0.25, 0.42]
	A_5	G	VG	G	(7.33, 8.67, 9.33)	[0.70, 0.89]
	A_6	F	F	F	(4.00, 5.00, 6.00)	[0.16, 0.32]
C_8	A_1	VP	P	P	(0.67, 1.67, 2.67)	[0.14, 0.25]
	A_2	G	G	MG	(6.33, 7.67, 8.67)	[0.76, 0.89]
	A_3	MG	MG	MG	(5.00, 7.00, 8.00)	[0.65, 0.82]
	A_4	F	MG	F	(4.33, 5.67, 6.67)	[0.55, 0.68]
	A_5	VP	VP	P	(0.33, 1.33, 2.33)	[0.11, 0.21]
	A_6	MG	MG	G	(5.67, 7.33, 8.33)	[0.70, 0.86]

Tab. 8. The relative closeness of each alternative to the ideal solution and the rank of each alternative

	Relative closeness to the ideal solution	Rank and priority
A_1	7.1492	4
A_2	7.5078	3
A_3	8.4432	2
A_4	6.4109	6
A_5	9.6641	1
A_6	6.5099	5

5. Conclusions

Multiple criteria decision making (MCDM), has widely use in the solution of real word decision making problems. By considering the fact that, in some cases, determining precisely the exact values of alternatives with respect to the criteria or / and the exact values for the weights of criteria, is difficult or impossible, so, the values of alternatives with respect to the criteria or / and the values of criteria weights are considered as fuzzy values (fuzzy numbers). Such that the conventional approaches for solving these MCDM problems tend to be less effective in dealing with the imprecise or vagueness nature of the linguistic assessment. In such conditions, the fuzzy MCDM methods are applied for solving MCDM problems with fuzzy data. In this paper, we propose a new fuzzy TOPSIS method based on left and right scores and then we apply proposed method for solving two numerical examples that the second numerical example is a real application related to the agent selection in the supply chain of an electronic company. As a result, the proposed method is practical for solving MCDM problems with fuzzy data and ranking alternatives in terms of their relative closeness to the ideal solution. Unlike many of MCDM methods, the proposed fuzzy TOPSIS method considers the decision makers (DMs) preference that is an advantage of it. Moreover, it seems that the proposed fuzzy TOPSIS method has a low computational volume and is flexible and easy to use.

It is expected that the fuzzy TOPSIS method have more potential applications in the near future.

References

[1] Hwang, C.L., Yoon, K., *Multiple Attribute Decision Making*: Methods and applications. Berlin: Springer, 1981.

[2] Wang, Y.M., Elhag, T.M.S., *Fuzzy TOPSIS Method Based on Alpha Level Sets with an Application to Bridge Risk Assessment.* Expert System with Applications, 31, 2006, pp. 309-319.

[3] Kahraman, C., Ates, N.Y., Cevik, S., Gulbay, M., Erdogan, A., *Hierarchical Fuzzy TOPSIS Model for Selection Among Logistics Information Technologies*. Journal of Enterprise Information Management, 20, 2007, 143-168.

[4] Boran, F.E., Genç, S., Kurt, M., Akay, D., *A Multi-Criteria Intuitionistic Fuzzy Group Decision Making for Supplier Selection with TOPSIS Method*. Expert Systems with Applications, 36, 2009, pp. 11363-11368.

[5] Amiri, M., Zandieh, M., Vahdani, B., Soltani, R., Roshanaei, V., *An Integrated Eigenvector–DEA–TOPSIS Methodology for Portfolio Risk Evaluation in the FOREX Spot Market*, Expert System with Applications, In Press, 2009.

[6] Chamodrakas, I., Alexopoulou, N., Martakos, D., *Customer Evaluation for Order Acceptance using a Novel Class of Fuzzy Methods Based on TOPSIS*. Expert Systems with Applications, 36, 2009, pp. 7409-7415.

[7] Sun, C. C., Lin, G. T. R., *Using Fuzzy TOPSIS Method for Evaluating the Competitive Advantages of Shopping Vebsites*. Expert Systems with Applications, In Press, 2009.

[8] Gumus, A.T., *Evaluation of Hazardous Waste Transportation Firms by using a two Step Fuzzy-AHP*

and TOPSIS Methodology. Expert Systems with Applications, 36, 2009, 4067-4074.

[9] Lin, M.C., Wang, C.C., Chen, M.S., Chang, C.A., *Using AHP and TOPSIS Approaches in Customer-Driven Product Design Process*. Computers in Industry, 59, 2008, 17–3.

[10] Onut, S., Soner, S., *Transshipment Site Selection using the AHP and TOPSIS Approaches under Fuzzy Environment.* Waste Management, 28, 2008, 1552–1559.

[11] Shih, H.S., *Incremental Analysis for MCDM with an Application to Group TOPSIS*. European Journal of Operational Research 186, 2008, 720–734.

[12] Deng, H., Yeh, C.H., Willis, R.J., *Inter-Company Comparison using Modified TOPSIS with Objective Weights*. Computers & Operations Research, 27, 2000, 963-973.

[13] Chen, C.T., *Extensions of the TOPSIS for Group Decision-Making Under Fuzzy Environment*. Fuzzy Sets and Systems, 114, 2000, 1-9.

[14] Opricovic, S., Tzeng, G.H., *Compromise Solution by MCDM Methods: A Comparative Analysis of VIKOR and TOPSIS*. European Journal of Operational Research, 156, 2004, pp. 445–455

[15] Wang, T.C., Lee, H.D., *Developing a Fuzzy TOPSIS Approach Based on Subjective Weights and Objective Weights.* Expert Systems with Applications, 36, 2009, 8980–8985.

[16] Ashtiani, B., Haghighirad, F., Makui, A., Montazer, G.A., *Extension of Fuzzy TOPSIS Method Based on Interval-Valued Fuzzy Sets*. Applied Soft Computing, 9, 2009, pp. 457–461.

[17] Jahanshahloo, G.R., Hosseinzadeh Lotfi, F., Davoodi, A.R., *Extension of TOPSIS for Decision-Making Problems with Interval data*: Interval efficiency. Mathematical and Computer Modelling, 49, 2009, pp. 1137-1142.

[18] Triantaphyllou, E., Lin, C.T., *Development and Evaluation of Five Fuzzy Multi Attribute Decision-making Methods.* International Journal of Approximate Reasoning, 14, 1996, pp. 281–310.

[19] Zadeh, L.A., Fuzzy sets. *Information and Control*, 8, 1965, 338–353.

[20] Chen, S.H., *Ranking Fuzzy Numbers with Maximization Set and Minimization Set*. Fuzzy Sets and Systems, 17, 1985, pp. 113-129.

[21] Chen, S.J., Hwang, C.L., *Fuzzy Multiple Attribute Decision Making*. Springer-Verlag, Berlin. 1992.

[22] Shipley, M.F., Korvin, D.K., Obit, R., *A Decision Making Model for Multi-Attribute Problems Incorporating Uncertainty and Bias Measures*. Computers and Operations Research,18, 1991, pp. 335–342.

[23] Deng, H., Yeh, C.H., Willis, R.J. *Inter-Company Comparison using Modified TOPSIS with Objective Weights*. Computers and Operations Research, 27, 2000, pp. 963–973.

Appendix A. Fuzzy arithmetic operations

Fuzzy sets are generalizations of crisp sets and were first introduced by Zadeh [19] as a way of representing imprecise or vagueness in real world. A fuzzy set is a collection of elements in a universe of information where the boundary of the set contained in the universe is ambiguous, vague and otherwise fuzzy. Each fuzzy set is specified by a membership function, which assigns to each element in the universe of discourse a value within the unit interval [0, 1]. The assigned value is called degree (or grade) of membership, which specifies the extent to which a given element belongs to the fuzzy set or is related to a concept. If the assigned value is 0, then the given element does not belong to the set. If the assigned value is 1, then the element totally belongs to the set. If the value lies within the interval (0, 1), then the element only partially belongs to the set. Therefore, any fuzzy set can be uniquely determined by its membership function.

Let X be the universe of discourse. A fuzzy set $\tilde{A}$ of the universe of discourse X is said to be convex if and only if for all x_1 and x_2 in X there always exists:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq Min\left(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right) \qquad \text{(a1)}$$

Where $\mu_{\tilde{A}}$ is the membership function of the fuzzy set $\tilde{A}$ and $\lambda \in$ [0, 1]. A fuzzy set $\tilde{A}$ of the universe of discourse X is said to be normal if there exists a $x_i \in X$ satisfying $\mu_{\tilde{A}}(x_i) = 1$. Fuzzy numbers are special cases of fuzzy sets that are both convex and normal. A fuzzy number is a convex fuzzy set, characterized by a given interval of real numbers, each with a grade of membership between 0 and 1.

The most commonly used fuzzy numbers are triangular and trapezoidal fuzzy numbers, whose membership functions are respectively defined as

$$\mu_{\tilde{A}_1}(x) = \begin{cases} (x-a)/(b-a) & a \leq x \leq b, \\ (d-x)/(d-b) & b \leq x \leq d, \\ 0 & otherwise. \end{cases} \qquad \text{(a2)}$$

$$\mu_{\tilde{A}_2}(x) = \begin{cases} (x-a)/(b-a) & a \leq x \leq b, \\ 1 & b \leq x \leq c, \\ (d-x)/(d-c) & c \leq x \leq d, \\ 0 & otherwise. \end{cases} \qquad \text{(a3)}$$

For brevity, triangular and trapezoidal fuzzy numbers are often denoted as (*a, b, d*) and (*a, b, c, d*). It is obvious that triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers with *b* = *c*.

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two positive triangular fuzzy numbers. Then basic fuzzy arithmetic operations on these fuzzy numbers are defined as

Addition:

$$\tilde{A} + \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Subtraction:

$$\tilde{A} - \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$$

Multiplication:

$$\tilde{A} * \tilde{B} \approx (a_1 b_1, a_2 b_2, a_3 b_3) \qquad \text{(a4)}$$

Division:

$$\tilde{A} \div \tilde{B} \approx \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right)$$

Appendix B. TOPSIS

TOPSIS (technique for order preference by similarity to an ideal solution) method is presented in Chen and Hwang [21], with reference to Hwang and Yoon [1]. The basic principle is that the chosen alternative should have the shortest distance from the ideal solution and the farthest distance from the negative-ideal solution. The TOPSIS procedure consists of the following steps:

Step 1: Calculate the normalized decision matrix. The normalized values $(y_{ij})_N$ are calculated as:

$$(y_{ij})_N = \frac{y_{ij}}{\sqrt{\sum_{i=1}^{n} y_{ij}^2}} \qquad i = 1, \ldots, n \quad , \quad j = 1, \ldots, m \qquad \text{(b1)}$$

In above formula, we consider y_{ij} as a rating of *i*th alternative with respect to *j*th criterion .

Step 2: Calculate the weighted normalized decision matrix. The weighted normalized values x_{ij} are calculated as:

$$x_{ij} = w_j \cdot (y_{ij})_N \qquad i = 1, \ldots, n \quad , \quad j = 1, \ldots, m \qquad \text{(b2)}$$

Where w_j is the weight of the *j*th criterion, and $\sum_{j=1}^{m} w_j = 1$.

Step 3: Determine the ideal and negative ideal solution.

$$I^* = \{x_1^*, \ldots, x_m^*\} = \left\{ (\underset{j}{Max}\ x_{ij} \mid j \in \Omega_b), (\underset{j}{Min}\ x_{ij} \mid j \in \Omega_c) \right\}$$
$$I^- = \{x_1^-, \ldots, x_m^-\} = \left\{ (\underset{j}{Min}\ x_{ij} \mid j \in \Omega_b), (\underset{j}{Max}\ x_{ij} \mid j \in \Omega_c) \right\} \qquad \text{(b3)}$$

Where Ω_b and Ω_c are defined as before.

Step 4: Calculate the separation measures, using the *n* dimensional Euclidean distance. The separation of each alternative from the ideal solution is given as

$$D_i^* = \sqrt{\sum_{j=1}^{m} (x_{ij} - x_j^*)^2} \qquad i = 1, \ldots, n \qquad \text{(b4)}$$

Similarly, the separation from the negative ideal solution is given as

$$D_i^- = \sqrt{\sum_{j=1}^{m} (x_{ij} - x_j^-)^2} \qquad i = 1, \ldots, n \qquad \text{(b5)}$$

Step 5: Calculate the relative closeness to the ideal solution. The relative closeness of the alternative A_i with respect to I^* is defined as

$$(RC)_i = \frac{D_i^-}{D_i^- + D_i^*} \qquad i = 1, \ldots, n \qquad \text{(b6)}$$

Step 6: Rank and prioritize the alternatives according to their relative closeness to the ideal solution.