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INFORMATION COVARIANCE MATRICES FOR MULTIVARIATE BURR III AND LOGISTIC DISTRIBUTIONS

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Abstract: *Main result of this paper is to derive the exact analytical expressions of information and covariance matrices for multivariate Burr III and logistic distributions. These distributions arise as tractable parametric models in price and income distributions, reliability, economics, Human population, some biological organisms to model agricultural population data and survival data. We showed that all the calculations can be obtained from one main moment multi dimensional integral whose expression is obtained through some particular change of variables. Indeed, we consider that this calculus technique for improper integral has its own importance.*

Keywords: *Gamma and Beta functions, Polygamma functions, Information and Covariance Matrices, Multivariate Burr III and Logistic models*

1. Introduction

In this paper the exact form of Fisher information matrices for multivariate Burr III and logistic distributions is determined. It is well-known that the information matrix is a valuable tool for derivation of covariance matrix in the asymptotic distribution of maximum likelihood estimations (MLE). In the univariate case for Pareto (IV) and Burr XII distributions, the Fisher information matrix is found by Brazauskas [4] and Watkins [12]. As discussed in Serfling [11], section 4, under suitable regularity conditions, the determinant of the asymptotic covariance matrix of (MLE) reaches and optimal lower bound for the volume of the spread ellipsoid of joint estimators. In multivariate case for Pareto (IV) and related distributions, the Fisher information matrices are found by Yari and Djafari[13] . The univariate logistic distribution has been studied rather extensively and, in fact, many of its developments through the years were moltivative to the normal distribution; for details see the handbook of Balakrishnan [3]. However, work on multivariate logistic distribution has been rather skimpy compared to the volumininous work that has been carried out on bivariate and multivariate normal distributions, Gumbel [8], Arnold [2], Johnson, Kotz and Balakrishnan[9], and Malik and Abraham [10]. For a broad discussion of logistic models and diverse applications see Malik and Abraham [10]. Burr III distribution arise as tractable parametric model have been formulated in the context of actuarial science reliability economiecs price and income distributions, Dagum [7], Burr [5] and Burr [6]. Logistic distribution arise as tractable parametric model have been formulated in human population, some biological organisms to model agricultural production data and survival data. This paper is organized as follows: Multivariate Burr III and logistic distribution are introduced and presented in section 2. Elements of the information and covariance matrix for multivariate Burr III distribution is derived in section 3. Elements of the information and covariance matrix for multivariate logistic distribution is derived in section 4. Conclusion is presented in section 5. Derivation of first and second derivatives of the log density function of multivariate Barr III distribution and calculation of its main moment integral are given in Appendices A and B . Derivation of first and second derivatives of the log-density of multivariate logistic distribution and calculation of its main moment integral are given in Appendices C and D.

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2 Multivariate Burr III and Logistic Distributions

The density function of the Burr III distribution is:

$$f_X(x) = \frac{ac\left(\frac{x-\mu)}{\theta}\right)^{-(c+1)}}{\theta\left(1+\left(\frac{x-\mu}{\theta}\right)^{-c}\right)^{\alpha+1}}, \quad x > \mu, \qquad (1)$$

where $-\infty < \mu < +\infty$ is the location parameter, $\theta > 0$ is the scale parameter, $c > 0$ and $\alpha > 0$ are the shape parameters which characterize the tail of the distribution.
The n-dimensional Burr III distribution is

$$f_n(\underline{x}) = \left(1 + \sum_{j=1}^{n} \left(\frac{x_j - \mu_j}{\theta_j}\right)^{-c_j}\right)^{-(\alpha+n)} \prod_{i=1}^{n} \frac{(\alpha + i - 1)c_i}{\theta_i} \left(\frac{x_i - \mu_i}{\theta_i}\right)^{-(c_i+1)}, \quad (2)$$

where $\underline{x} = [x_1, \ldots, x_n]$, $x_i > \mu_i, c_i > 0, -\infty < \mu_i < +\infty$, $\alpha > 0$, $\theta_i > 0$ for $i = 1, \ldots, n$. One of the main properties of this distribution is that, the joint density of any subset of the components of a multivariate Burr III random vector is again of the form (2) [9]. The density of the logistic distribution is

$$f_X(\underline{x}) = \frac{\alpha}{\theta} e^{-\left(\frac{x-\mu}{\theta}\right)} \left(1 + e^{-\left(\frac{x-\mu}{\theta}\right)}\right)^{-(\alpha+1)}, \quad x > \mu, \quad (3)$$

where $-\infty < \mu < +\infty$ is the location parameter, $\theta > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter.
The n-dimensional logistic distribution is

$$f_n(\underline{x}) = \left(1 + \sum_{j=1}^{n} e^{-\left(\frac{x_j - \mu_j}{\theta_j}\right)}\right)^{-(\alpha+n)} \prod_{i=1}^{n} \frac{(\alpha+i-1)}{\theta_i} e^{-\left(\frac{x_i - \mu_i}{\theta_i}\right)}, \quad (4)$$

where $\underline{x} = [x_1, \ldots, x_n]$, $x_i > \mu_i$, $\alpha > 0$, $-\infty < \mu_i < +\infty$ and $\theta_i > 0$ for $i = 1, \ldots, n$. The joint density of any subset of the components of a multivariate logistic random vector is again of the form (4) [9].

3. Information Matrix for Multivariate Burr III

Suppose X is a random vector with the probability density function where $\Theta = (\theta_1, \theta_2, \ldots \theta_K)$. The information matrix $I(\Theta)$ is the $K \times K$ matrix with elements

$$I_{ij}(\Theta) = -E_{\Theta}\left[\frac{\partial^2 \ln f_{\Theta}(X)}{\partial \theta_i \partial \theta_j}\right], \quad i, j = 1, \ldots, K \quad (5)$$

For the multivariate Burr III, we have $\Theta = (\mu_1, \ldots \mu_n, \theta_1, \ldots, \theta_n, c_1, \ldots, c_n, \alpha)$. In order to make the multivariate Burr III distribution a regular family (in terms of maximum likelihood estimation), we assume that vector μ is known and, without loss of generality, equal to 0. In this case information matrix is $(2n+1) \times (2n+1)$. Thus, further treatment is based on the following multivariate density function.

$$f_n(\underline{x}) = \left(1 + \sum_{j=1}^{n} \left(\frac{x_j}{\theta_j}\right)^{-c_j}\right)^{-(\alpha+n)} \times \prod_{i=1}^{n} \frac{(\alpha + i - 1)c_i}{\theta_i} \left(\frac{x_i}{\theta_i}\right)^{-(c_i+1)}, \quad x_i > 0. \quad (6)$$

The log-density function is:

$$\ln f_n(\underline{x}) = \sum_{i=1}^{n} \left(\ln(\alpha + i - 1) - \ln\theta_i + \ln c_i - (c_i + 1)\ln\left(\frac{x_i}{\theta_i}\right)\right) - (\alpha + n)\ln\left(1 + \sum_{j=1}^{n} \left(\frac{x_j}{\theta_j}\right)^{-c_j}\right). \quad (7)$$

Since the information matrix $I(\Theta)$ is symmetric, it is enough to find elements $I_{ij}(\Theta)$, where $1 \le i \le j \le 2n+1$. The first and second partial derivatives of the above expression are given in the Appendix A. In order to determine the information matrix and score functions, we need to find:

$$E\left[\ln\left(1 + \sum_{j=1}^{n} \left(\frac{X_j}{\theta_j}\right)^{-c_j}\right)\right], \quad E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l} \left(\frac{X_k}{\theta_k}\right)^{-c_k}\right] \text{ and}$$

evaluation of the required orders partial derivatives of the last expectation at the required points.

3-1. Main Strategy to Obtain Expressions of the Expectations

Derivation of these expressions are based on the following strategy: first, we derive an analytical expression for the following integral:

$$E\left[\prod_{i=1}^{n} \left(\frac{X_i}{\theta_i}\right)^{r_i}\right] = \int_{\circ}^{+\infty} \ldots \int_{\circ}^{+\infty} \prod_{i=1}^{n} \left(\frac{x_i}{\theta_i}\right)^{r_i} f_n(\underline{x})\, d\underline{x}, \quad (8)$$

and then we show that all other expressions can be found easily from this. We consider this derivation as one of the main contributions of this work. This derivation is given in the Appendix B. The result as the following :

$$E\left[\prod_{i=1}^{n}\left(\frac{X_i}{\theta_i}\right)^{r_i}\right] = \int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{i=1}^{n}\left(\frac{x_i}{\theta_i}\right)^{r_i} f_n(\underline{x})\, d\underline{x}, \quad (9)$$

$$= \frac{\Gamma(\alpha+\sum_{i=1}^{n}\frac{r_i}{c_i})\prod_{i=1}^{n}\Gamma(1-\frac{r_i}{c_i})}{\Gamma(\alpha)}$$

$$= \sum_{i=1}^{n}\frac{r_i}{c_i} < \alpha, \frac{r_i}{c_i} < 1,$$

where Γ is the usual Gamma function,

$$\Gamma_{r_l r_k}\left(\alpha+\sum_{i=1}^{n}\frac{r_i}{c_i}\right) = \frac{\partial^2\Gamma\left(\alpha+\sum_{i=1}^{n}\frac{r_i}{c_i}\right)}{\partial r_k \partial r_l}, \quad 1\le l \ , \ k \le n,$$

$$\Psi^{(n)}(z) = \frac{d_n}{dz^n}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right), \ z>0$$

$$\frac{\partial^{(m+n)}}{\partial r_l^m \partial r_k^n}\left(\frac{\Gamma_{r_l r_k}(z)}{\Gamma(z)}\right) = \Psi^{(m+n)}(z), \ z>0,$$

where the integers n, m are nonnegative. Specifically, we use digamma $\Psi(z)=\Psi^{(\circ)}(z)$, trigamma $\Psi'(z)$ and $\Psi_{r_l r_k}(z)$ functions, Abramowitz [1] and Brazauskas [4]. To confirm the regularity of $\ln f_n(\underline{x})$ and evaluation of the expected Fisher information matrix, we take expectations of the first and second order partial derivatives of (7). All other expressions can be derived from this main result. Taking derivative with respect to α, from the both sides of the relation

$$1 = \int_0^{+\infty} f_n(\underline{x}) d\underline{x}, \quad (10)$$

leads us to

$$E[\ln(1+\sum_{j=1}^{n}(\frac{X_j}{\theta_j})^{-c_j}] = \sum_{i=1}^{n}\frac{1}{\alpha+i+1}. \quad (11)$$

From the relation (9), for a pair of (ι,k) we have

$$\varphi(r_\iota, r_k) = E\left[\left(\frac{X_l}{\theta_l}\right)^{r_l}\left(\frac{X_k}{\theta_k}\right)^{r_k}\right] = \frac{\Gamma(\alpha+\frac{r_l}{c_l}+\frac{c_k}{c_k})\Gamma(-\frac{r_l}{c_l}+1)\Gamma(-\frac{r_k}{c_k}+1)}{\Gamma(\alpha)}. \quad (12)$$

From relation (12), when $r_k = 0$ we obtain

$$\zeta(r_l) = E\left[\left(\frac{X_l}{\theta_l}\right)^{r_l}\right] = \frac{\Gamma(\alpha+\frac{r_l}{c_l})\Gamma(-\frac{r_l}{c_l}+1)}{\Gamma(\alpha)}, \quad (13)$$

evaluating this expectation at $r_l=-c_l$, $r_l=-2c_l$ and the relation (12) at $(r_l=-c_l, r_k=-c_k)$, we obtain:

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{c_l}\right] = \frac{1}{\alpha-1}, \quad (14)$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-2c_l}\right] = \frac{2}{(\alpha-1)(\alpha-2)},$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\left(\frac{X_k}{\theta_k}\right)^{-c_k}\right] = \frac{1}{(\alpha-1)(\alpha-2)}. \quad (15)$$

Evaluating the required orders partial derivatives of (13) and (12) at the required points, we have

$$E\left[\ln\left(\frac{X_l}{\theta_l}\right)\right] = \frac{[\Psi(\alpha)-\Gamma'(1)]}{c_l},$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\ln\left(\frac{X_l}{\theta_l}\right)\right] = \frac{[\Psi(\alpha-1)-\Gamma'(2)]}{c_l(\alpha-1)}, \quad (16)$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\ln^2\left(\frac{X_l}{\theta_l}\right)\right] = \frac{[\Psi^2(\alpha-1)+\Psi'(\alpha-1)-2\Psi(\alpha-1)\Gamma'(2)+\Gamma''(2)]}{c_l^2(\alpha-1)},$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-2c_l}\ln^2\left(\frac{X_l}{\theta_l}\right)\right] = \frac{[2\Psi^2(\alpha-2)+2\Psi'(\alpha-2)-2\Psi(\alpha-2)\Gamma'(3)+\Gamma''(3)]}{c_l^2(\alpha-1)(\alpha-2)},$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\ln\left(\frac{X_l}{\theta_l}\right)\right] = \frac{[2\Psi^2(\alpha-2)+\Gamma'(3)]}{c_l(\alpha-1)(\alpha-2)},$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\left(\frac{X_k}{\theta_k}\right)^{-c_k}\ln\left(\frac{X_k}{\theta_k}\right)\right] = \frac{[\Psi_{r_l}(\alpha-2)+\Gamma'(2)]}{c_k(\alpha-1)(\alpha-2)},$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\left(\frac{X_k}{\theta_k}\right)^{-c_k}\ln\left(\frac{X_l}{\theta_l}\right)\ln\left(\frac{X_k}{\theta_k}\right)\right] = \frac{[-\Gamma'(2)\Psi_{r_l}(\alpha-2)+\Psi_{r_k}(\alpha-2)]}{c_l c_k(\alpha-1)(\alpha-2)} + \frac{[(\Gamma'(2))^2+\Psi_{r_l r_k}(\alpha-2)]}{c_l c_k(\alpha-1)(\alpha-2)}.$$

From (14), (15) and (16) with α replaced by $(\alpha+1)$ and $(\alpha+2)$ in (6), we can show that

$$E\left[\frac{\left(\frac{X_l}{\theta_l}\right)^{-c_l}}{\left(1+\sum_{j=1}^{n}\left(\frac{X_i}{\theta_j}\right)^{-c_j}\right)}\right] = \frac{\alpha}{\alpha+n}E_{\alpha+1}\left[\left(\frac{X_l}{\theta_l}\right)^{-c_l}\right] = \frac{1}{\alpha+n}, \quad (17)$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-2c_l}}{(1+\sum_{j=1}^{n}(\frac{X_i}{\theta_j})^{-c_j})^2}\right]$$

$$=\frac{\alpha(\alpha+1)}{(\alpha+n)(\alpha+n+1)}E_{\alpha+2}\left[(\frac{X_l}{\theta_l})^{-2c_l}\right]$$

$$=\frac{2}{(\alpha+n)(\alpha+n+1)},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-c_l}\ln^2(\frac{X_l}{\theta_l})}{1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}}\right]=$$

$$\frac{[\Psi^2(\alpha)+\Psi'(\alpha)-2\Psi(\alpha)\Gamma'(2)+\Gamma''(2)]}{c_1^2(\alpha+n)},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-c_l}\ln(\frac{X_l}{\theta_l})}{1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}}\right]=\frac{[\Psi(\alpha)-\Gamma'(2)]}{c_l(\alpha+n)},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-2c_l}\ln(\frac{X_l}{\theta_l})}{\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^2}\right]=\frac{[2\Psi(\alpha)-\Gamma'(3)]}{c_l(\alpha+n)(\alpha+n+1)},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-2c_l}\ln^2(\frac{X_l}{\theta_l})}{\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^2}\right]=$$

$$\frac{[2\Psi^2(\alpha)+\Gamma''(3)-2\Gamma'(3)\Psi(\alpha)+2\Psi'(\alpha)]}{c_1^2(\alpha+n)(\alpha+n+1)},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-c_l}(\frac{X_k}{\theta_k})^{-c_k}\ln(\frac{X_k}{\theta_k})}{\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^2}\right]=\frac{[\Psi_{rk}(\alpha)-\Gamma'(2)]}{c_k(\alpha+n)(\alpha+n+1)},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^{-c_l}(\frac{X_k}{\theta_k})^{-c_k}\ln(\frac{X_l}{\theta_l})\ln(\frac{X_k}{\theta_k})}{\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^2}\right]=$$

$$\frac{[\Psi_{rl\,rk}(\alpha)-\Gamma'(2)(\Psi_{rl}(\alpha)+\Psi_{rk}(\alpha))+(\Gamma'(2))^2]}{c_l c_k(\alpha+n)(\alpha+n+1)}.$$

3-2. Expectations of the Score Functions

The expectations of the first three derivations of the first order follow immediately from the corresponding results for their three corresponding parameters and we obtain:

$$E\left[\frac{\partial \ln f_n(\underline{X})}{\partial \alpha}\right]=\sum_{i=1}^{n}\frac{1}{\alpha+i-1}-E\left[\ln(1+\sum_{j=1}^{n}(\frac{X_j}{\theta_j})^{-c_j})\right]=0\ ,\quad(18)$$

$$E\left[\frac{\partial \ln f_n(\underline{X})}{\partial \theta_l}\right]=\frac{c_\iota}{\theta_l}-\left(\frac{(\alpha+n)c_l}{\theta_\iota}\right)\times E\left[\frac{(\frac{x_l}{\theta_l})^{-c_l}}{1+\sum_{j=1}^{n}(\frac{X_i}{\theta_j})^{-c_j}}\right]=0\ ,\quad(19)$$

$$E\left[\frac{\partial \ln f_n(\underline{X})}{\partial c_l}\right]=\frac{1}{c_\iota}-E\left[\ln\left(\frac{X_l}{\theta_l}\right)\right]+(\alpha+n)\times E\left[\frac{(\frac{x_\iota}{\theta_\iota})^{-c_\iota}\ln(\frac{X_\iota}{\theta_\iota})}{1+\sum_{j=1}^{n}(\frac{X_i}{\theta_j})^{-c_j}}\right]=0.\quad(20)$$

3-3. The Expected Fisher Information Matrix

Main strategy is again based on the integral (9) which is presented in the Appendix B. After some tedious algebraic simplifications, the following expressions can be obtained:

$$I_x(\alpha)=\sum_{i=1}^{n}\frac{1}{(\alpha+i-1)^2}\ ,\quad(21)$$

$$I_x(\theta_1,\alpha)=\frac{c_\iota}{\theta_\iota(\alpha+n)}\ ,\quad(22)$$

$$I_x(c_1,\alpha)=-\frac{1}{c_\iota(\alpha+n)}[\Psi(\alpha)-\Gamma'(2)],\quad \iota=1,..,n,\quad(23)$$

$$I_x(\theta_\iota)=\frac{c_\iota^2(\alpha+n-1)}{\theta_\iota^2(\alpha+n+1)}\ ,\ \iota=1,...,n\ ,\quad(24)$$

$$I_x(c_1)=-\frac{1}{c_\iota^2}[1+\Gamma''(2)-2\Psi(\alpha)\Gamma'(2)+\Psi^2(\alpha)\Psi'(\alpha)]-\frac{2}{c_\iota^2(\alpha+n+1)}$$

$$[-\Gamma'(3)\Psi(\alpha)+\Psi^2(\alpha)+\Psi'(\alpha)+\frac{1}{2}\Gamma''(3)]\ ,\ l=1,...n.$$

$$I_x(\theta_l,\theta_k) = -\frac{c_l c_k}{\theta_l\theta_k(\alpha+n+1)} \quad , \quad k \neq l \quad , (26)$$

$$I_x(c_1,c_k) = -\frac{[(\Gamma'(2))^2 - \Gamma'(2)(\Psi_{r_k}(\alpha)) + \Psi_{r_l}(\alpha)) + \Psi_{r_l r_k}(\alpha)}{c_l c_k(\alpha+n+1)}, \quad (27)$$
$$k \neq l,$$

$$I_x(\theta_l,c_k) = \frac{[-\Gamma'(2)+\Psi_{r_k}(\alpha)]}{\theta_l c_k(\alpha+n+1)} \quad , \quad k \neq l, \quad (28)$$

$$I_x(\theta_l,c_l) = \frac{[\Gamma'(2)\Psi(\alpha)]}{\theta_l} + \frac{[2\Psi(\alpha)-\Gamma'(3)]}{\theta_l(\alpha+n+1)}, \quad (29)$$
$$l = 1,\ldots,n.$$

Thus the information matrix, $I_{Burr\_III}(\Theta)$, for the multivariate Burr III $(0,\theta,c,\alpha)$ distribution is:

$$I_{Burr\_III}(\Theta) = \begin{bmatrix} I(\theta_l,\theta_k) & I(\theta_l,c_k) & I(\theta_l,\alpha) \\ I(\theta_l,c_k) & I(c_l,c_k) & I(c_l,\alpha) \\ I(\theta_l,\alpha) & I(c_l,\alpha\ ) & I(\alpha) \end{bmatrix}. \quad (30)$$

3-4. Covariance Matrix for Multivariate Burr III

Since the joint density of any subset of the components of a multivariate Burr III random vector is again a multivariate Burr III [9], we can calculate the expectation:

$$E\left[\left(\frac{X_l-\mu_l}{\theta_l}\right)^{m_l}\left(\frac{X_k-\mu_k}{\theta_k}\right)^{m_k}\right] = \quad (31)$$
$$\int_0^\infty\int_0^\infty\left(\frac{x_l-\mu_l}{\theta_l}\right)^{m_l}\left(\frac{x_k-\mu_k}{\theta_k}\right)^{m_k} f_{X_lX_k}(x_l,x_k)dx_l dx_k$$
$$= \frac{\Gamma(\alpha+\frac{m_l}{c_l}+\frac{m_k}{c_k})\Gamma(1-\frac{m_l}{c_l})\Gamma(1-\frac{m_k}{c_k})}{\Gamma(\alpha)},$$
$$1-\frac{m_l}{c_l}>0 \ , \ 1-\frac{m_k}{c_k}>0 \ , \ \alpha+\frac{m_l}{c_l}+\frac{m_k}{c_k}>0.$$

Evaluating this expectation at $(m_l=1,m_k=0)$, $(m_l=0,m_k=1)$, $(m_l=1,m_k=1)$ and $(m_l=m,\ m_k=0)$ we obtain:

$$E(X_l) = \mu_l + \frac{\theta_l}{\Gamma(\alpha)}[\Gamma(\alpha+\frac{1}{c_l})\Gamma(I-\frac{1}{c_l})], \quad (32)$$

$$E[X_k] = \mu_k + \frac{\theta_k}{\Gamma(\alpha)} + [\Gamma(\alpha+\frac{1}{c_k})\Gamma(1-\frac{1}{c_k})], \quad (33)$$
$$E[X_l X_k] = \mu_k E[X_l] + \mu_l E[X_k] - \mu_l\mu_k$$
$$+\frac{\theta_l\theta_k}{\Gamma(\alpha)}[\Gamma(\alpha+\frac{1}{c_k}+\frac{1}{c_k})\Gamma(1-\frac{1}{c_k}-\frac{1}{c_l})], \quad (34)$$

$$E[X_l^{\,m}] = \frac{\theta_l^m}{\Gamma(\alpha)}[\Gamma(\alpha+\frac{m}{c_l})\Gamma(1-\frac{m}{c_l})], \quad (35)$$

$$\sigma^2_{X_l} = \frac{\theta_l^2}{\Gamma^2(\alpha)}$$
$$\times\left[\Gamma(\alpha+\frac{2}{c_1})\Gamma(1-\frac{2}{c_l})\Gamma(\alpha)-\Gamma^2(1-\frac{1}{c_l})\Gamma^2(\alpha+\frac{1}{c_l})\right], \quad (36)$$
$$1-\frac{2}{c_l}>0,$$

$$\mathrm{cov}[X_l,X_k] = \frac{\theta_l\theta_k}{\Gamma^2(\alpha)}\Gamma(1-\frac{1}{c_l})\Gamma(1-\frac{1}{c_k})$$
$$[\Gamma(\alpha+\frac{1}{c_l}+\frac{1}{c_k})\Gamma(\alpha) \quad (37)$$
$$-\Gamma(\alpha+\frac{1}{c_k})\Gamma(\alpha+\frac{1}{c_l})],$$
$$1\le l<k\le n, \quad k=2,\ldots,n.$$

4. Information Matrix for Logistic Distribution

For the multivariate logistic distribution, we have $\Theta=(\mu_1,\ldots,\mu_n,\theta_1,\ldots,\theta_n,\alpha)$.

In order to make the multivariate logistic distribution a regular family (in terms of maximum likelihood estimation), we assume that vector μ is known and, without loss of generality equal to 0 . In this case information matrix is of order $(n+1)\times(n+1)$. Thus, further treatment is based on the following multivariate density function

$$f_n(\underline{x}) = \left(1+\sum_{j=1}^n e^{-(\frac{x_j}{\theta_j})}\right)^{-(\alpha+n)} \prod_{i=1}^n \frac{(\alpha+i-1)}{\theta_i} e^{-(\frac{x_i}{\theta_i})}. \quad (38)$$

Thus, the log- density function is :

$$\ln f_n(\underline{x}) = \sum_{i=1}^n[\ln(\alpha+i-1] - \ln\theta_i] -$$
$$\sum_{i=1}^n(\frac{x_i}{\theta_i}) - (\alpha+n)\ln(1+\sum_{j=1}^n e^{-(\frac{x_j}{\theta_j})}]. \quad (39)$$

Since the information matrix $I(\Theta)$ is symmetric it is enough to find elements $I_{ij}(\Theta)$, where $1\le i\le j\le n+1$. The first and second partial derivatives of the above expression are given in the Appendix C. Looking at these expressions, we see that to determine the

expression of the information matrix and score functions, we need to find the following expectations

$$E\left[\ln\left(1+\sum_{j=1}^{n} e^{-(\frac{x_j}{\theta_j})}\right)\right], E\left[\left(e^{-(\frac{X_l}{\theta_l})}\right)^{r_l}\left(e^{-(\frac{X_k}{\theta_k})}\right)^{r_k}\right],$$

and evaluation of the required orders partial derivatives of the last expectation at the required points.

4-1. Main Strategy to Obtain Expressions of the Expectations

Derivation of these expressions are based on the following strategy: first, we derive an analytical expressionfor the following integral :

$$E\left[\prod_{i=1}^{n}\left(e^{-(\frac{X_i}{\theta_i})}\right)^{r_i}\right]=$$

$$\int_{\circ}^{+\infty}\ldots\int_{\circ}^{+\infty}\prod_{i=1}^{n}\left(e^{(\frac{x_i}{\theta_i})}\right)^{r_i} f_n(\underline{x})\,d\underline{x}, \quad (40)$$

and then, we show that all the other expressions can be found easily from it. This derivation is given in the Appendix D. The result is the following:

$$E\left[\prod_{i=1}^{n}\left(e^{-(\frac{X_i}{\theta_i})}\right)^{r_i}\right]=\int_{\circ}^{+\infty}\ldots\int_{\circ}^{+\infty}\prod_{i=1}^{n}\left(e^{(\frac{x_i}{\theta_i})}\right)^{r_i} f_n(\underline{x})\,d\underline{x}=$$

$$\frac{\Gamma(\alpha-\sum_{i=1}^{n} r_i)\prod_{i=1}^{n}\Gamma(1+r_i)}{\Gamma(\alpha)}, \quad (41)$$

$$\sum_{i=1}^{n} r_i<\alpha, r_i>-1.$$

Taking derivative with respect to α, from the both sides of the relation

$$1=\int_0^{+\infty} f_n(\underline{x})d\underline{x}, \quad (42)$$

leads us to

$$E\left[\ln\left(1+\sum_{j=1}^{n} e^{-(\frac{x_j}{\theta_j})}\right)\right]=\sum_{i=1}^{n}\frac{1}{\alpha+i-1}. \quad (43)$$

From relation (41), for a pair of (l,k) we have

$$\varphi(r_l,r_k)=E\left[\left(e^{-(\frac{X_l}{\theta_l})}\right)^{r_l}\left(e^{-(\frac{X_k}{\theta_k})}\right)^{r_k}\right]=$$

$$\frac{\Gamma(\alpha-r_l-r_k)\Gamma(r_k+1)\Gamma(r_l+1)}{\Gamma(\alpha)}. \quad (44)$$

From relation (44), at $r_k=0$ we obtain

$$E\left[\left(e^{-(\frac{X_l}{\theta_l})}\right)^{r_l}\right]=\frac{\Gamma(\alpha-r_l)\Gamma(r_l+1)}{\Gamma(\alpha)}, \quad (45)$$

and evaluating this expectation at $r_l=1$, we obtain

$$E\left[\left(e^{-(\frac{X_l}{\theta_l})}\right)\right]=\frac{1}{\alpha-1}. \quad (46)$$

Differentiating first and second order of (45) with respect to r_l and replacing for $r_l=0$, $r_l=1$ and $r_l=2$, we obtain the following relations:

$$E\left[\left(\frac{X_l}{\theta_l}\right)\right]=\Psi(\alpha)-\Gamma'(1), \quad (47)$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)\left(e^{-\left(\frac{X_l}{\theta_l}\right)}\right)\right]=\left[\frac{\Psi(\alpha-1)-\Gamma'(2)}{(\alpha-1)}\right], \quad (48)$$

and

$$E\left[\left(\frac{X_l}{\theta_l}\right)^2\left(e^{-\left(\frac{X_l}{\theta_l}\right)}\right)\right]=$$

$$\left[\frac{\Psi^2(\alpha-1)-\Gamma'(2)\Psi(\alpha-1)+\Psi'(\alpha-1)+\Gamma''(2)}{(\alpha-1)}\right], \quad (49)$$

$$E\left[\left(\frac{X_l}{\theta_l}\right)^2 e^{-2\left(\frac{X_l}{\theta_l}\right)}\right]=$$

$$\left[\frac{\Gamma''(3)-2\Gamma'(3)\Psi(\alpha-2)+2\Psi^2(\alpha-2)+2\Psi'(\alpha-2)}{(\alpha-1)(\alpha-2)}\right]. \quad (50)$$

From relation (44),

$$\frac{\partial}{\partial r_l\partial r_k}\varphi(r_l=1,r_k=1)=$$

$$E\left[(\frac{X_l}{\theta_l})(\frac{X_k}{\theta_k})(e^{-(\frac{X_l}{\theta_l})})(e^{-(\frac{X_k}{\theta_k})})\right] \quad (51)$$

$$\left[\frac{[\Gamma'(2)\left(\Gamma'(2)-\Psi_{r_k}(\alpha-2)+\Psi_{r_l}(\alpha-2)\right)]}{(\alpha-1)(\alpha-2)}\right]$$

$$+\frac{\Psi_{r_lr_k}(\alpha-2)}{(\alpha-1)(\alpha-2)}.$$

With α replaced by $(\alpha+1)$ and $(\alpha+2)$ in (38) we obtain

$$E\left[\frac{(e^{-(\frac{X_l}{\theta_l})})}{(1+\sum_{j=1}^{n}e^{-(\frac{X_j}{\theta_j})})}\right]=\frac{\alpha}{\alpha+n}E_{\alpha+1}[e^{-(\frac{X_l}{\theta_l})}] \quad (52)$$
$$=\frac{1}{\alpha+n},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})e^{-(\frac{X_l}{\theta_l})}}{(1+\sum_{j=1}^{n}e^{-(\frac{X_j}{\theta_j})})}\right]=\frac{\alpha}{\alpha+n}E_{\alpha+1}[(\frac{X_l}{\theta_l})e^{-(\frac{X_l}{\theta_l})}] \quad (53)$$
$$=\frac{\Psi(\alpha)\Gamma'(2)}{\alpha+n},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^2e^{-(\frac{X_l}{\theta_l})}}{(1+\sum_{j=1}^{n}e^{-(\frac{X_j}{\theta_j})})^2}\right]= \quad (54)$$
$$\frac{\Psi^2(\alpha)+\Psi'(\alpha)+\Gamma''(2)-2\Gamma'(2)\Psi(\alpha)}{\alpha+n},$$

$$E\left[\frac{(\frac{X_l}{\theta_l})^2e^{-2(\frac{X_l}{\theta_l})}}{(1+\sum_{j=1}^{n}e^{-(\frac{X_j}{\theta_j})})^2}\right]=$$
$$\frac{[2\Psi^2(\alpha)+2\Gamma'(3)\Psi(\alpha)+2\Psi'(\alpha)+\Gamma''(3)]}{(\alpha+n)(\alpha+n+1)}, \quad (55)$$

$$E\left[\frac{(\frac{X_k}{\theta_k})(\frac{X_l}{\theta_l})e^{-(\frac{X_l}{\theta_l})}e^{-(\frac{X_k}{\theta_k})}}{(1+\sum_{j=1}^{n}e^{-(\frac{X_j}{\theta_j})})^2}\right]= \quad (56)$$
$$\frac{\Gamma'(2)[\Gamma'(2)-\Psi_{r_k}(\alpha)-\Psi_{r_l}(\alpha)]+\Psi_{r_lr_k}(\alpha)}{(\alpha+n)(\alpha+n+1)}.$$

4-2. Expectations of the Score Functions

The expectations of the first two derivations of the first order follow immediately from the corresponding results for their two corresponding parameters and we obtain:

$$E\left[\frac{\partial \ln f_{n(\underline{X})}}{\partial\alpha}\right]=\sum_{i=1}^{n}\frac{1}{\alpha+i-1} \quad (57)$$
$$-E\left[\ln\left(1+\sum_{j=1}^{n}e^{-(\frac{X_j}{\theta_j})}\right)\right]=0,$$

$$E\left[\frac{\partial \ln f_n(\underline{X})}{\partial\theta_l}\right]=\frac{1}{\theta_l}+\frac{1}{\theta_l}[\Psi(\alpha)-\Gamma'(1)]$$
$$-\frac{1}{\theta_l}[\Psi(\alpha)-\Gamma'(2)]=0. \quad (58)$$

4-3. The Expected Fisher Information Matrix

Main strategy is again based on the integral (40) which is presented in the Appendix D.
After some tedious algebraic simplifications, the following expressions can be obtained.

$$I_x(\alpha)=\sum_{i=1}^{n}\frac{1}{(\alpha+i-1)^2}, \quad (59)$$

$$I_x(\theta_l,\alpha)=\frac{1}{\theta_l(\alpha+n)}[\Psi(\alpha)-\Gamma'(2)], \quad (60)$$
$$l=1,\ldots,n,$$

$$I_x(\theta_l)=\frac{(\alpha+n-1)}{\theta_l^2(\alpha+n+1)}$$
$$[\Psi^2(\alpha)-2\Gamma'(2)\Psi(\alpha)+\Gamma''(2)+\Psi'(\alpha)], \quad (61)$$
$$+\frac{1}{\theta_l^2}-2\left[\frac{\Gamma'(2)\Psi(\alpha)}{\theta_l^2(\alpha+n+1)}\right],\ l=1,\ldots,n,$$

$$I_x(\theta_l,\theta_k)=$$
$$-\left[\frac{\Gamma'(2)[\Gamma'(2)-\Psi_{r_k}(\alpha)-\Psi_{r_l}(\alpha)]+\Psi_{r_lr_k}(\alpha)}{\theta_l\theta_k(\alpha+n+1)}\right], \quad (62)$$
$$k\neq l.$$

Thus the information matrix , $I_{ML}(\Theta)$, for the multivariate logistic $(0,\theta,\alpha)$ distribution is

$$I_{ML}(\Theta)=\begin{bmatrix}I(\theta_l,\theta_k) & I(\theta_l,\alpha)\\ I(\theta_l,\alpha) & I(\alpha)\end{bmatrix}. \quad (63)$$

4-4. Covariance Matrix for Multivariate Logistic

Since the joint density of any subset of the components of a multivariate logistic random vector is again multivariate logistic distribution (4) [9], we can use the relation (44), (47) and obtain

$$\frac{\partial}{\partial_{r_l}\partial_{r_k}}\varphi(r_l=0,r_k=0)=E[X_lX_k]$$
$$=\theta_l\theta_k[(\Gamma'(1))^2-\Gamma'(1)(\Psi_{r_k}(\alpha)$$
$$-\Psi_{r_l}(\alpha))+\Psi_{r_lr_k}(\alpha)], \quad (64)$$
$$k\neq l$$

$$E[X_l]=\theta_l[\Psi(\alpha)-\Gamma'(1)],\ l=1,\ldots,n, \quad (65)$$

$$E[X_k]=\theta_k[\Psi(\alpha)-\Gamma'(1)],\ k=1,\ldots,n. \quad (66)$$

From second order derivative of relation (44), we have

$$\frac{\partial}{\partial r_l^2}\varphi(r_l=0, r_k=0) = E[X_l^2] = \theta_l^2(\Gamma''(1)) - 2\Gamma'(1)\Psi(\alpha) + \Psi^2(\alpha) + \Psi'(\alpha)], \quad l=1,\ldots,n, \tag{67}$$

$$\operatorname{cov}[X_l, X_k] = \theta_l\theta_k[-\Gamma'(1)(\Psi_{r_k}(\alpha) + \Psi_{r_l}(\alpha)) + \Psi_{r_l r_k}(\alpha) - 2\Psi(\alpha) - \Psi^2(\alpha)], \quad k \neq l, \tag{68}$$

$$\sigma_{X_l}^2 = \theta_l^2[\Gamma''(1) - (\Gamma'(1))^2 + \Psi'(\alpha)], l=1,\ldots,n. \tag{69}$$

5. Conclusion

In this paper we obtained the exact forms of Fisher information and covariance matrices for multivariate Burr III and multivariate logistic distributions. We showed that in both distributions, all of the expectations can be obtained from two main moment multi dimensional integrals which have been considered and whose expression is obtained through some particular change of variables. A short method of obtaining some of the expectations as a function of α is used. To confirm the regularity of the multivariate densities, we showed that the expectations of the score functions are equal to 0 ,

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Appendix A: Expressions of the Derivatives

In this Appendix, we give detail, the expressions for the first and second derivatives of $\ln f_n(\underline{x})$, where, $f_n(\underline{x})$ is the multivariate Burr III density function (6), which are needed for obtaining the expression of the information matrix:

$$\frac{\partial \ln f_n(\underline{x})}{\partial \alpha} = \sum_{i=1}^{n}\frac{1}{\alpha+i-1} - \ln\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right), \tag{1}$$

$$\frac{\partial \ln f_n(\underline{x})}{\partial \theta_l} = \frac{c_l}{\theta_l} - \frac{(\alpha+n)c_l}{\theta_l}\frac{(\frac{x_l}{\theta_l})^{-c_l}}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}, \quad l=1,\ldots,n, \tag{2}$$

$$\frac{\partial \ln f_n(\underline{x})}{\partial c_l} = \frac{1}{c_l} - \ln(\frac{x_l}{\theta_l}) + (\alpha+n) \times \frac{(\frac{x_l}{\theta_l})^{-c_l}\ln(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}, \quad l=1,\ldots,n, \tag{3}$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial \alpha^2} = -\sum_{j=1}^{n}\frac{1}{(\alpha+i-1)^2}, \tag{4}$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial \theta_l \partial \alpha} = \frac{-c_l}{\theta_l}\frac{(\frac{x_l}{\theta_l})^{-c_l}}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}, \quad l=1,\ldots,n, \tag{5}$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial c_l \partial \alpha} = \frac{(\frac{x_l}{\theta_l})^{-c_l}\ln(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}, \quad l=1,\ldots,n, \tag{6}$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial \theta_l^2} = \frac{-c_l}{\theta_l^2} + \frac{(\alpha+n)(1-c_l)c_l}{\theta_l^2} \frac{(\frac{x_l}{\theta_l})^{-c_l}}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}$$

$$+\left(\frac{(\alpha+n)c_l^2}{\theta_l^2}\right)\frac{(\frac{x_l}{\theta_l})^{-2c_l}}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})^2}, \qquad (7)$$

$$l = 1,...,n,$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial c_l^2} = \frac{-1}{c_l^2} - (\alpha+n)\frac{(\frac{x_l}{\theta_l})^{-c_l}\ln^2(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})} +$$

$$(\alpha+n)\frac{(\frac{x_l}{\theta_l})^{-2c_l}\ln^2(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})^2}, \qquad (8)$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial \theta_k \partial \theta_l} = \left(\frac{(\alpha+n)c_l c_k}{\theta_l \theta_k}\right) - \frac{(\frac{x_l}{\theta_l})^{-c_l}(\frac{x_k}{\theta_k})^{-c_k}}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})^2}, \qquad (9)$$

$$k \neq l$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial c_k \partial \theta_l} = -\left(\frac{(\alpha+n)c_l}{\theta_l}\right) - \frac{(\frac{x_l}{\theta_l})^{-c_l}(\frac{x_k}{\theta_k})^{-c_k}\ln(\frac{x_k}{\theta_k})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})^2}, \qquad (10)$$

$$k \neq l$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial c_k \partial c_l} = (\alpha+n)\frac{(\frac{x_l}{\theta_l})^{-c_l}(\frac{x_k}{\theta_k})^{-c_k}\ln(\frac{x_k}{\theta_k})\ln(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})^2}, \qquad (11)$$

$$k \neq l,$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial c_k \partial \theta_l} = \frac{1}{\theta_l} - \left(\frac{\alpha+n}{\theta_l}\right) - \frac{(\frac{x_l}{\theta_l})^{-c_l}}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}$$

$$+\left(\frac{(\alpha+n)c_i}{\theta_l}\right)\frac{(\frac{x_l}{\theta_l})^{-c_l}\ln(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})}$$

$$-\left(\frac{(\alpha+n)c_i}{\theta_l}\right)\frac{(\frac{x_l}{\theta_l})^{-2c_l}\ln(\frac{x_l}{\theta_l})}{(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j})^2} \qquad l = 1,...,n.$$

Appendix B. Expression of the Main Integral

This Appendix gives one of the main results of this paper which is the derivation of the expression of the following integral:

$$E\left[\prod_{i=1}^{n}\left(\frac{X_i}{\theta_i}\right)^{r_i}\right] = \int_0^{+\infty}\cdots\int_0^{+\infty}\prod_{i=1}^{n}\left(\frac{x_i}{\theta_i}\right)^{r_i} f_n(\underline{x})\,d\underline{x}, \quad (1)$$

where, $f_n(\underline{x})$ is the multivariate Burr III density function (6). This derivation is done in the following steps:

First consider the following one dimensional integral :

$$C_1 = \int_0^{+\infty}\frac{\alpha c_1}{\theta_1}(\frac{x_1}{\theta_1})^{r_1}(\frac{x_1}{\theta_1})^{-(c_1+1)}\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^{-(\alpha+n)} dx_1$$

$$= \int_0^{+\infty}\frac{\alpha c_1}{\theta_1}(\frac{x_1}{\theta_1})^{r_1}(\frac{x_1}{\theta_1})^{-(c_1+1)}\left(1+\sum_{j=1}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^{-(\alpha+n)}$$

$$\times\left(1+\frac{(\frac{x_1}{\theta_1})^{-c_1}}{1+\sum_{j=2}^{n}(\frac{x_j}{\theta_j})^{-c_j}}\right)^{-(\alpha+n)} d\,x_1. \qquad (2)$$

Note that, goings first line to second line is just a factorizing and rewriting the last term of the integrand. After many reflections on the links between Burr families and Gamma and Beta functions, we found that the following change of variable:

$$\left(1+\frac{(\frac{x_1}{\theta_1})^{-c_1}}{1+\sum_{j=2}^{n}(\frac{x_j}{\theta_j})^{-c_j}}\right) = \frac{1}{1-t}, \quad 0 < t < 1, \qquad (3)$$

simplifies this integral and guides us to the following result.

$$C_1 = \frac{\alpha\,\Gamma(1-\frac{r_1}{c_1})\Gamma(\alpha+n-1+\frac{r_1}{c_1})}{\Gamma(\alpha+n)}$$

$$\times\left(1+\sum_{j=2}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^{-(\alpha+n)-r_1+1}. \qquad (4)$$

Then we consider the following similar expression:

$$C_2 = \int_0^{+\infty}\frac{c_2\alpha(\alpha+1)\Gamma(1-\frac{r_l}{c_l})\Gamma(\alpha+n-1+\frac{r_l}{c_l})}{\theta_2\Gamma(\alpha+n)}(\frac{x_2}{\theta_2})^{r_2}$$

$$(\frac{x_2}{\theta_2})^{-(c_2+1)}\left(1+\sum_{j=2}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^{-(\alpha+n)-\frac{r_1}{c_1}+1} dx_2,$$

$$C_2 = \int_0^{+\infty} \frac{c_2\alpha(\alpha+1)\Gamma(1-\frac{r_l}{c_l})\Gamma(\alpha+n-1+\frac{r_l}{c_l})}{\theta_2\Gamma(\alpha+n)}(\frac{x_2}{\theta_2})^{r_2}$$

$$(\frac{x_2}{\theta_2})^{r_2}(\frac{x_2}{\theta_2})^{-(c_2+1)}\left(1+\sum_{j=3}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^{-(\alpha+n)-\frac{r_1}{c_1}+1}$$

$$\left(1+\frac{(\frac{x_2}{\theta_2})^{-c_2}}{1+\sum_{j=3}^{n}(\frac{x_j}{\theta_j})^{-c_j}}\right)^{-(\alpha+n)-\frac{r_1}{c_1}+1} d\,x_2,$$

and again using the following change of variable:

$$\left(1+\frac{(\frac{x_2}{\theta_2})^{-c_2}}{1+\sum_{j=3}^{n}(\frac{x_j}{\theta_j})^{-c_j}}\right) = \frac{1}{1-t},$$

we obtain :

$$C_2 = \frac{\alpha(\alpha+1)\Gamma(1-\frac{r_1}{c_1})\Gamma(1-\frac{r_2}{c_2})\Gamma(\alpha+n-2+\frac{r_l}{c_1}+\frac{r_1}{c_1})}{\Gamma(\alpha+n)}$$

$$\times\left(1+\sum_{j=3}^{n}(\frac{x_j}{\theta_j})^{-c_j}\right)^{-(\alpha+n)-\frac{r_1}{c_1}-\frac{r_2}{c_2}+2}.$$

Continuing this method, finally, we obtain the general expression:

$$C_n = E\left[\prod_{i=1}^{n}(\frac{X_i}{\theta_i})^{r_i}\right] = \frac{\Gamma(\alpha+\sum_{i=1}^{n}\frac{r_i}{c_i})\prod_{i=1}^{n}\Gamma(1-\frac{c_i}{c_i})}{\Gamma(\alpha)},$$

$$\sum_{i=1}^{n}\frac{r_i}{c_i} < \alpha, 1-\frac{r_i}{c_i} > 0.$$

Appendix C. Expressions of the Derivatives

In this Appendix, we give in detail the expressions for the first and second derivatives of $\ln f_n(\underline{x})$, where, $f_n(\underline{x})$ is the multivariate logistic density function (38), which are needed for obtaining the expression of the information matrix:

$$\frac{\partial \ln f_n(\underline{x})}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{\alpha+i-1} - \ln\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right), \quad (1)$$

$$\frac{\partial \ln f_n(\underline{x})}{\partial\theta_l} = -\frac{1}{\theta_l}+\frac{x_l}{\theta_l^2}-\frac{(\alpha+n)}{\theta_l}\frac{(\frac{x_l}{\theta_l})e^{-(\frac{x_l}{\theta_l})}}{\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right)}, \quad (2)$$

$$l = 1,\ldots,n,$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial\alpha^2} = -\sum_{j=1}^{n}\frac{1}{(\alpha+i-1)^2}, \quad (3)$$

$$\frac{\partial \ln f_n(\underline{x})}{\partial\theta_l\partial\alpha} = -\frac{(\frac{x_l}{\theta_l})e^{-(\frac{x_l}{\theta_l})}}{\theta_l\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right)}, \quad l = 1,\ldots,n, \quad (4)$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial\theta_k\partial\theta_l} =$$

$$\frac{(\alpha+n)}{\theta_l\theta_k}\frac{(\frac{x_l}{\theta_l})(\frac{x_k}{\theta_k})e^{-(\frac{x_l}{\theta_l})}e^{-(\frac{x_k}{\theta_k})}}{\theta_l\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right)^2}, \quad k \neq 1, \quad (5)$$

$$\frac{\partial^2 \ln f_n(\underline{x})}{\partial\theta_l^2} = \quad (6)$$

$$-\frac{1}{\theta_l^2}-\frac{2x_l}{\theta_l^3}+\frac{2(\alpha+n)}{\theta_l^2}\frac{(\frac{x_l}{\theta_l})e^{-(\frac{x_l}{\theta_l})}}{\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right)}$$

$$-\frac{(\alpha+n)}{\theta_l^2}\frac{(\frac{x_l}{\theta_l})^2e^{-(\frac{x_l}{\theta_l})}}{\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right)}$$

$$+\frac{(\alpha+n)}{\theta_l^2}\frac{(\frac{x_l}{\theta_l})^2e^{-2(\frac{x_l}{\theta_l})}}{\left(1+\sum_{j=1}^{n}e^{-(\frac{x_j}{\theta_j})}\right)^2}. \quad l = 1,\ldots,n,$$

Appendix D. Expression of the Main Integral

This Appendix gives the second main result of this paper which is the derivation of the expression of the following integral

$$E\left[\prod_{i=1}^{n}\left(e^{-(\frac{X_i}{\theta_i})}\right)^{r_i}\right] = \quad (1)$$

$$\int_0^{+\infty}\ldots\int_0^{+\infty}\prod_{i=1}^{n}\left(e^{-(\frac{x_i}{\theta_i})}\right)^{r_i} f_n(\underline{x})d\underline{x},$$

where, $f_n(\underline{x})$ is the multivariate logistic density function (38). This derivation is done in the following steps :
First consider the following one dimensional integral :

$$C_n = \int_0^{+\infty} \frac{(\alpha+n-1)}{\theta_n} \left(e^{-(\frac{x_n}{\theta_n})} \right)^{r_n} e^{-(\frac{x_n}{\theta_n})} \times \left(1+\sum_{j=1}^{n} e^{-(\frac{x_j}{\theta_j})} \right)^{-(\alpha+n)} dx_n$$

$$= \int_0^{+\infty} \frac{(\alpha+n-1)}{\theta_n} \left(e^{-(\frac{x_n}{\theta_n})} \right)^{r_n} e^{-(\frac{x_n}{\theta_n})} \left(1+\sum_{j=1}^{n-1} e^{-(\frac{x_j}{\theta_j})} \right)^{-(\alpha+n)} \times \left(1+\frac{e^{-(\frac{x_n}{\theta_n})}}{1+\sum_{j=1}^{n-1} e^{-(\frac{x_j}{\theta_j})}} \right)^{-(\alpha+n)} dx_n . \quad (2)$$

Note that, goings from first line to second line is just a factorizing and rewriting the last term of the integrand. After many reflections on the links between logistic function and Gamma and Beta functions we found that the following change of variable:

$$\left(1+\frac{e^{-(\frac{x_n}{\theta_n})}}{1+\sum_{j=1}^{n-1} e^{-(\frac{x_j}{\theta_j})}} \right) = \frac{1}{1-t} \quad 0<t<1, \quad (3)$$

simplifies this integral and guides us to the following result

$$C_n = \frac{\Gamma(r_n+1)\Gamma(\alpha+n-r_n-1)}{\Gamma(\alpha+n-1)} \times \left(1+\sum_{j=1}^{n-1} e^{-(\frac{x_j}{\theta_j})} \right)^{-(\alpha+n)+r_n+1} . \quad (4)$$

Then we consider the following similar expression:

$$C_{n-1} = \frac{\Gamma(r_n+1)\Gamma(\alpha+n-r_n-1)}{\Gamma(\alpha+n-1)} \int_0^{+\infty} \frac{(\alpha+n-2)}{\theta_{n-1}} \left(e^{-(\frac{x_{n-1}}{\theta_{n-1}})} \right)^{r_{n-1}} \times e^{-(\frac{x_{n-1}}{\theta_{n-1}})} \left(1+\sum_{j=1}^{n-1} e^{-(\frac{x_j}{\theta_j})} \right)^{-(\alpha+n)+r_n+1} dx_{n-1}$$

$$= \frac{\Gamma(r_n+1)\Gamma(\alpha+n-r_n-1)}{\Gamma(\alpha+n-1)} \times \int_0^{+\infty} \frac{(\alpha+n-2)}{\theta_{n-1}} \left(e^{-(\frac{x_{n-1}}{\theta_{n-1}})} \right)^{r_{n-1}} \times e^{-(\frac{x_{n-1}}{\theta_{n-1}})} \left(1+\sum_{j=1}^{n-2} e^{-(\frac{x_j}{\theta_j})} \right)^{-(\alpha+n)+r_n+1} \times \left(1+\frac{e^{-(\frac{x_{n-1}}{\theta_{n-1}})}}{1+\sum_{j=1}^{n-2} e^{-(\frac{x_j}{\theta_j})}} \right)^{-(\alpha+n)+r_n+1} dx_{n-1}, \quad (6)$$

and again using the following change of variable:

$$\left(1+\frac{e^{-(\frac{x_{n-1}}{\theta_{n-1}})}}{1+\sum_{j=1}^{n-2} e^{-(\frac{x_j}{\theta_j})}} \right) = \frac{1}{1-t} ,$$

we obtain:

$$C_{n-1} = \frac{\Gamma(r_{n-1}+1)\Gamma(r_n+1)\Gamma(\alpha+n-r_n-r_{n-1}-1)}{\Gamma(\alpha+n-2)} \times \left(1+\sum_{j=1}^{n-2} e^{-(\frac{x_j}{\theta_j})} \right)^{-(\alpha+n)+r_n+r_{n-1}+1} . \quad (7)$$

Continuing this method, finally, we obtain the general expression:

$$C_1 = E\left[\prod_{i=1}^{n} \left(e^{-(\frac{x_i}{\theta_i})^{r_i}} \right)^{r_i} \right] \quad (8)$$

$$= \frac{\Gamma(\alpha-\sum_{i=1}^{n} r_i) \prod_{i=1}^{n} \Gamma(1+r_i)}{\Gamma(\alpha)}, \quad \sum_{i=1}^{n} r_i < \alpha, \ r_i > -1.$$