IUST International Journal of Engineering Science,
Vol. 19, No.5-1, 2008, Page 137-142

QUANTITATIVE NON-DIAGONAL REGULATOR DESIGN FOR UNCERTAIN MULTIVARIABLE SYSTEM WITH HARD TIME-DOMAIN CONSTRAINTS

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Abstract: *In this paper a non-diagonal regulator, based on the QFT method, is synthesized for an uncertain MIMO plant whose output and control signals are subjected to hard time-domain constraints. This procedure includes the design of a non-diagonal pre-controller based on a new simple approach, followed by the sequential design of a diagonal QFT controller. We present a new formulation for the latter stage, which shows the role of off-diagonal elements in the design procedure. A numerical example is given to illustrate the effectiveness of the proposed method.*

Keywords*:* *MIMO, QFT, non-diagonal, control, uncertain*

1. Introduction

In the past few years, several control techniques to design non-diagonal controllers for uncertain MIMO systems have been proposed by Yaniv[1], Boje[2], Y.H. Chang and J.C. Chang [3] and Garcia-Sanz and Egana[4]. Some of these approaches such as [2] have focused on the design of off-diagonal elements of controllers based on the reduction of interaction between the elements. As shown in [1], improving the diagonal dominance is not necessarily the best criterion for designing a non-diagonal controller, instead reducing the bandwidth of the controller is a more reasonable criterion.

The work in [1] has concentrated only on plant behavior at high frequencies. It requires using the initially developed plant in an n-stage sequential procedure. Then, the uncertainty of the next equivalent SISO systems will not be included. Thus, another simple procedure to contribute all uncertainties is required. A new approach is proposed in this paper to address this problem.

One of the applied problems in uncertain MIMO systems which has considered before by some researchers such as Franchek[5] is the design of robust regulator under certain hard time-domain constraints on output and control signals of the systems in response to the step disturbances. This problem has been solved before in MIMO QFT framework based on diagonal controller. Since non-diagonal elements of the controller can improve the ability of the design; this problem will be discussed in more details. Off-diagonal elements of this regulator are synthesized based on the new proposed method and the diagonal elements are synthesized based on Yaniv's approach [6].

In this paper, we propose a new simple approach for designing non-diagonal controllers within the framework of the sequential MIMO QFT. This involves first, the design of a non-diagonal controller, followed by the design of a standard diagonal controller to achieve stability and performance specifications.

Also, we present the formulation for the design of non-diagonal regulator to meet simultaneously hard time-domain constraints on both the outputs and the control signals and show the role of non-diagonal elements in the design procedure. An example is included to illustrate the new formulation.

The arrangement of the paper is as follows. In section II, the problem is stated. In section III, design of non-diagonal elements of the regulator is explained. In section IV, the new formulation for the design of diagonal elements of the regulator is developed and section V illustrates the method by its application to an example. Section VI concludes the paper.

2. The Problem Statement

The problem statement without loss of generality is given for a 2×2 system for simplicity. Consider the system shown in Fig. 1, where **P** is a 2×2 LTI plant belonging to a set $\{\boldsymbol{P}\}$, $\boldsymbol{d}$ is a step disturbances vector belonging to a given set $\{\boldsymbol{d}\}$, α and $\boldsymbol{\beta}$ are 2×1 constant vectors which introduce output and control signal constraints.

Paper first received May. 10, 2007 and in revised form Sep. 26, 2009.

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Design the non-diagonal regulator, $\boldsymbol{G}$, such that for all $\boldsymbol{P} \in \{\boldsymbol{P}\}$;

- The system is stable; and
- For all $\boldsymbol{d} \in \{\boldsymbol{d}\}$, the plant output $\boldsymbol{y} = [y_1 \quad y_2]^T$ and control signal $\boldsymbol{u} = [u_1 \quad u_2]^T$ are bounded by

$$|y_k(t)| \le \alpha_k \ , |u_k(t)| \le \beta_k \quad k = 1,2 \tag{1}$$

We assume that $\mu = d_2 / d_1$ is a given constant.

3. Design of Non-Diagonal Elements of *G*

Similar to Yaniv's method [1], we assume that $\boldsymbol{G} = \boldsymbol{G_n G_d}$, where

$$\boldsymbol{G}_d = diag(g_1, g_2) \quad , \boldsymbol{G}_n = \begin{bmatrix} 1 & g_{12} \\ g_{21} & 1 \end{bmatrix}$$

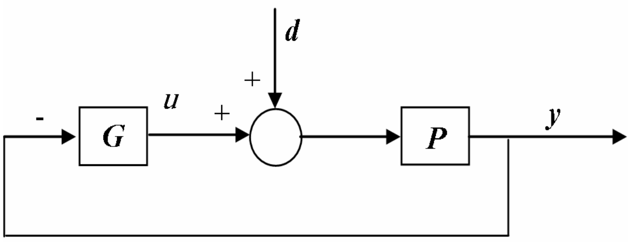

Fig. 1. A MIMO feedback system with disturbances at the inputs to the plant

As we know from the basics of QFT, the effect of uncertainty is more effective in low frequencies. Then the minimization of dimensions of the plant templates in all frequencies is the motivation of this method. By using the $\Delta-$norm of a matrix $\mathbf{A}$ which is defined in [7] as:

$$\|\boldsymbol{A}\|_{\Delta} = \max_{i,j} |a_{ij}| \tag{2}$$

We can define a function whose values represent the template sizes. This function is defined as

$$U(\omega) = \sum_{i=1}^{k} c_i u(\omega) \tag{3}$$

where $u(\omega) = \|\boldsymbol{PG}_n - \boldsymbol{P}_0\boldsymbol{G}_n\|$ and $\boldsymbol{P_0}$ is the nominal plant. The weights c_i are used for tuning. The function $u(\omega)$ as illustrated in Fig. 2 measures the maximum distances between all uncertain plants to the nominal plant at some frequency ω. The function U sums up and weights them for all frequencies. Here, we select

$c_i = 1/\omega_i$

The best selection of these coefficients can be further investigated.

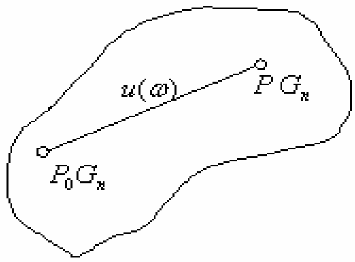

Fig. 2. Template of $\boldsymbol{PG_n}$ in some frequency ω.

4. Design of the Diagonal Elements of G

In Fig. 1, we can write:

$$\boldsymbol{y} = (\boldsymbol{I} + \boldsymbol{PG})^{-1} \boldsymbol{Pd} \tag{4}$$

Since we assumed that $\boldsymbol{G} = \boldsymbol{G_n G_d}$, we have

$$\boldsymbol{y} = (\boldsymbol{I} + \boldsymbol{PG_n G_d})^{-1} \boldsymbol{PG_n G_n^{-1} d} \tag{5}$$

Now, if we consider $\boldsymbol{PG_n}$ as a new plant and $\boldsymbol{G_n^{-1} d}$ as a new disturbance vector, then we can use the standard sequential MIMO QFT procedure [6] as follows:

$$(\boldsymbol{PG_n})^{-1} = \boldsymbol{\Lambda} + \boldsymbol{B} \tag{6}$$

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1/q_{11} & 0 \\ 0 & 1/q_{22} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 1/q_{12} \\ 1/q_{21} & 0 \end{bmatrix}$$

$$\boldsymbol{y} = \boldsymbol{T}^D \boldsymbol{d}' \quad , \boldsymbol{T}^D = \begin{bmatrix} t_{11}^D & t_{12}^D \\ t_{21}^D & t_{22}^D \end{bmatrix},$$

$$\boldsymbol{d}' = \boldsymbol{G_n^{-1} d} = (1/\Delta)[1 - g_{12}\mu \quad \mu - g_{21}]^T d$$

$$\Delta = \det(\boldsymbol{G_n}) \quad , \quad \boldsymbol{d} = [d_1 \quad d_2]^T \quad , \quad d_2 = \mu d_1 = \mu d$$

$\boldsymbol{T}^D$=matrix transfer function from outputs to disturbances. Its elements are:

$$t_{11}^D = \frac{q_{11}}{1 + g_1 q_{11}} \left\{ 1 - \frac{t_{21}^D}{q_{12}} \right\} \qquad t_{12}^D = \frac{q_{11}}{1 + g_1 q_{11}} \left\{ - \frac{t_{22}^D}{q_{12}} \right\}$$

$$t_{21}^D = \frac{q_{22}}{1 + g_2 q_{22}} \left\{ - \frac{t_{11}^D}{q_{21}} \right\} \qquad t_{22}^D = \frac{q_{22}}{1 + g_2 q_{22}} \left\{ 1 - \frac{t_{12}^D}{q_{21}} \right\}$$

By substituting the latter equations in the above equation for t_{11}^D, t_{12}^D we will get the following equation:

$$y_1 = t_{11}^D d_1' + t_{12}^D d_2' = t_{11}^D [\frac{1}{\Delta}(1 - g_{12}\mu) d + t_{12}^D [\frac{1}{\Delta}(\mu - g_{21}) d]$$

or

$$y_1(s) = \frac{q_{11}}{1 + g_1 q_{11}} \cdot \frac{1}{\Delta} \left[(1 - g_{12}\mu) - \frac{1}{q_{12}} \left\{ t_{21}^D (1 - g_{12}\mu) + t_{22}^D (\mu - g_{21}) \right\} \right]$$

Using transformation lemma, $|y_1(j\omega)| \le \alpha_1$, then we get the following inequality:

$$\left|\frac{q_{11}}{1+g_1q_{11}}\cdot\frac{1}{\Delta}\right|\left|(1-g_{12}\mu)-\frac{1}{q_{12}}\left[t_{21}^D(1-g_{12}\mu)+t_{22}^D(\mu-g_{21})\right]\right|\le\alpha_1 \qquad (7)$$

If we use the following triangular inequality:

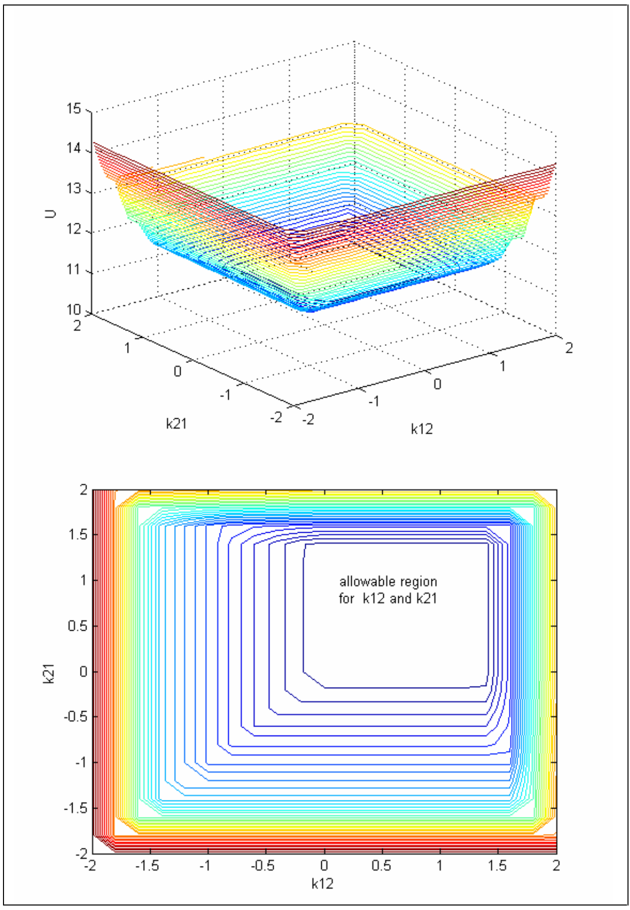

Fig. 3. Contours of constant uncertainty.

$$\left||a|-|b|\right|\le|a-b|\le|a|+|b|$$

then by mathematical operations;

$$\left|\frac{q_{11}}{1+g_1q_{11}}\right|\cdot\left|\left|\frac{1-g_{12}\mu}{\Delta}\right|+\frac{\alpha_2}{|q_{12}|}\right|\le\alpha_1 \qquad (8)$$

A similar inequality is derived from $|y_2(t)|\le\alpha_2$ as follows:

$$\left|\frac{q_{22}}{1+g_2q_{22}}\right|\cdot\left|\left|\frac{\mu-g_{21}}{\Delta}\right|+\frac{\alpha_1}{|q_{21}|}\right|\le\alpha_2 \qquad (9)$$

For translating time-domain constraints on control signals, we substitute:

$$u_1=-g_1y_1-g_{12}g_2y_2 \qquad (10)$$

By substituting y_1, y_2 from the closed loop transfer functions between outputs and disturbances in (10) yields:

$$u_1(s)=u_{11}(s)d(s)+u_{12}(s)d(s)=T_1(s)d(s) \qquad (11)$$

where
T_1=transfer function from d to u_1

$$u_{11}=-\frac{g_1}{(1+g_1q_{11})\Delta}\left[(1-g_{12}\mu)-\frac{1}{q_{21}}\left\{t_{21}^D(1-g_{12}\mu)+t_{22}^D(\mu-g_{21})\right\}\right]$$

$$u_{12}=-\frac{g_{21}q_{22}}{(1+g_2q_{22})\Delta}\left[(\mu-g_{21})-\frac{1}{q_{21}}\left\{t_{11}^D(1-g_{12}\mu)+t_{12}^D(\mu-g_{21})\right\}\right]$$

From transformation lemma [8] and specifications (1), g_1 and g_2 should be designed to satisfy the following inequalities in all frequencies:

$$\begin{aligned}&f_1(g_1)+|g_{12}|\cdot f_2(g_2)\le\beta_1\\&|g_{21}|\cdot f_1(g_1)+f_2(g_2)\le\beta_2\end{aligned} \qquad (12)$$

Where:

$$f_1(g_1)=a\left|\frac{g_1q_{11}}{1+g_1q_{11}}\right| \qquad a=\left|\frac{1-g_{12}\mu}{\Delta}\right|+\frac{\alpha_2}{|q_{12}|} \qquad (13\text{-}14)$$

$$f_2(g_2)=b\left|\frac{g_2q_{22}}{1+g_2q_{22}}\right| \qquad b=\left|\frac{\mu-g_{21}}{\Delta}\right|+\frac{\alpha_1}{|q_{21}|}$$

From the inequalities (8), (9) and (12), the role of the non-diagonal part $\boldsymbol{G_n}$ becomes clear: it allows lower bounds for the designed g_1 and g_2, which results in a bandwidth lower than the one achievable with a diagonal controller.
We can not use the inequalities (12) in computations of bounds in QFT toolbox [9], because g_1 and g_2 appear in both of them. We have to solve (12) at first and find the limits on $f_1(g_1)$ and $f_2(g_2)$ for all uncertainties. After this step, we can use QFT toolbox and compute the bounds on nominal open loop transfer functions in Nicholes chart.
It must be noted that the optimum non-diagonal controller which has designed in the first step can not be optimum in the second step.

5. An Illustrative Example

A. An Uncertain Plant

Consider the feedback system shown in Fig. 1, where the uncertain plant family is given by:

$$\boldsymbol{P}=\frac{1}{s^2}\begin{bmatrix}k_1 & k_2\\ k_2 & k_1\end{bmatrix};\ k_1\in[2\ \ 4],\ k_2\in[1.0\ \ 1.8] \qquad (15)$$

We assume that the disturbances are the same, i.e. $\mu=1$.
The performance is as follows

$$|y_1(t)|\le 0.5,\ |y_2(t)|\le 0.5,\ |u_1(t)|\le 2,\ |u_2(t)|\le 2$$

Gain margin=6 db

This example is different from [1] only in specifications.

B. Design of $\boldsymbol{G_n}$

The non-diagonal elements of matrix transfer function $\boldsymbol{G_n}$ were chosen as:

$$g_{12} = \frac{k_{12}}{s^2 + 3s}, g_{21} = \frac{k_{21}}{s^2 + 3s}$$

and $c_i = 1/\omega_i$. Fig. 3 shows $U(\omega)$ as a function of the off-diagonal gain of g_{12} element (x-axis) and its off-diagonal gain of g_{21} element (y-axis). A solution is:

$$\boldsymbol{G}_n = \begin{bmatrix} 1 & \frac{1}{s^2+3s} \\ \frac{1}{s^2+3s} & 1 \end{bmatrix}$$

Fig. 3 shows that k_{12}, k_{21} can be selected in the range $[-0.2 \;\; 1.4]$. We can do some adjustments for improving some specifications such as, stability, bandwidth, and so on. We select them here for better loop shaping in the next step (loop shaping for diagonal elements). This selection depends on the problem.

C. Design of $\boldsymbol{G_d}$

Figs. 4 and 5 show the QFT bounds and the open loops frequency response for the diagonal design. After the loop shaping process, the diagonal controller elements are designed as:

$$g_1(s) = \frac{4.265}{(s^2/476.4^2 + 1.734s/476.4 + 1)}$$
$$g_2(s) = \frac{2.83}{(s/27.65 + 1)(s/191.3 + 1)}$$

There are three group bounds in Fig. 4 and 5, i.e. robust stability, robust performances. Upper bounds are shown dotted curves and lower bounds are shown full curves. The intersection region of these bounds is the allowable region for loop shaping of the controllers. If there is no intersection region for some frequencies, then loop shaping is not possible and there is no solution. We can select another set of non-diagonal elements as discussed in section B, or if it is possible, time-domain constraints on the output and the control signals must be changed.

D. Result and Discussions

Fig. 6 to 9 shows the simulation results of the system with the designed controller. As it is seen, all desired specifications are satisfied. Both input disturbances are the unit step functions. The results show that the regulators are somewhat conservative. Templates of $(P)_{11}$ and $(PG_n)_{11}$ at $\omega = 0.1$ rad/s are shown in Figs. 10 and 11 typically. We can see templates of other elements of P and PG_n. These figures show that the size of the template in Fig. 11 is lower than Fig. 10. This difference is not considerable here but it may gets more values in other examples. The smaller size of the plant template, the lower bandwith of the open loop is achived.

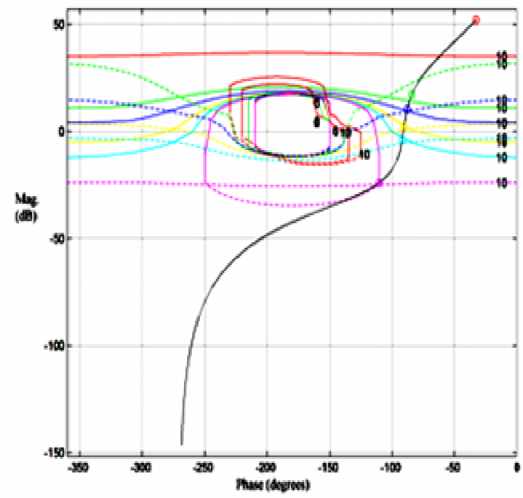

Fig. 4. Upper and lower bounds and $L_{01}(j\omega)$ in Nichols chart.

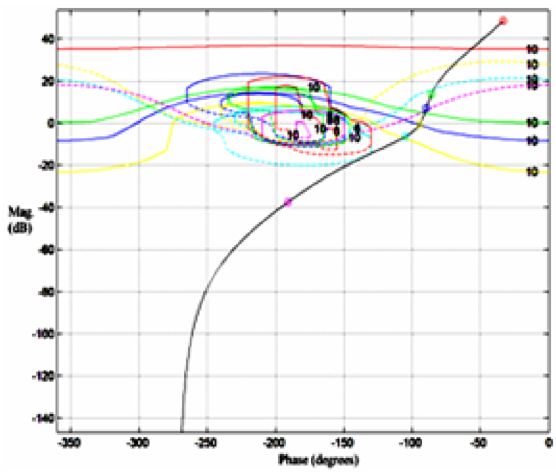

Fig. 5. Upper and lower bounds and $L_{02}(j\omega)$ in Nichols chart.

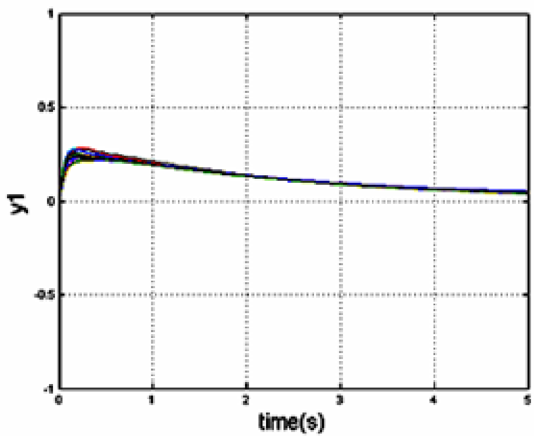

Fig. 6. Time-domain simulation of output signal y_1 to unit step disturbances.

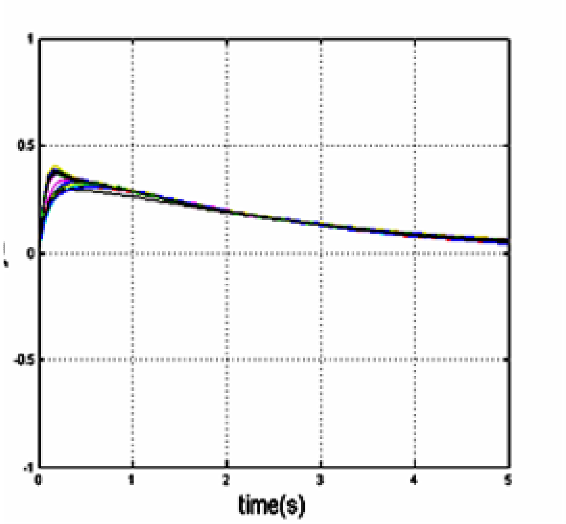

Fig.7. Time-domain simulation of output signal y_2 to unit step disturbances.

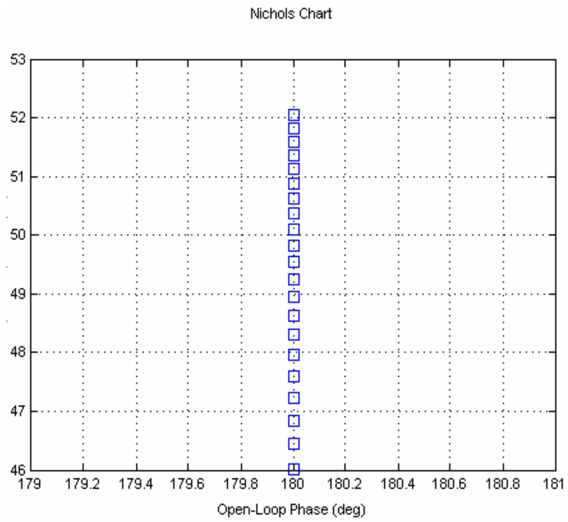

Fig. 10. Template of $(P)_{11}$ at $\omega = 0.1$ rad/s.

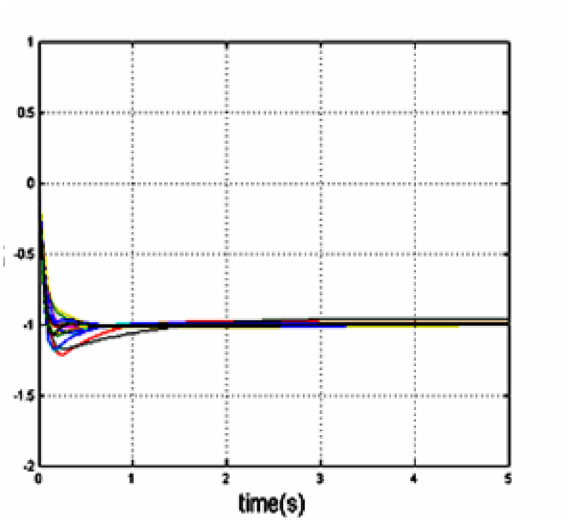

Fig.8. Time-domain simulation of control signal u_1 due to unit step disturbances.

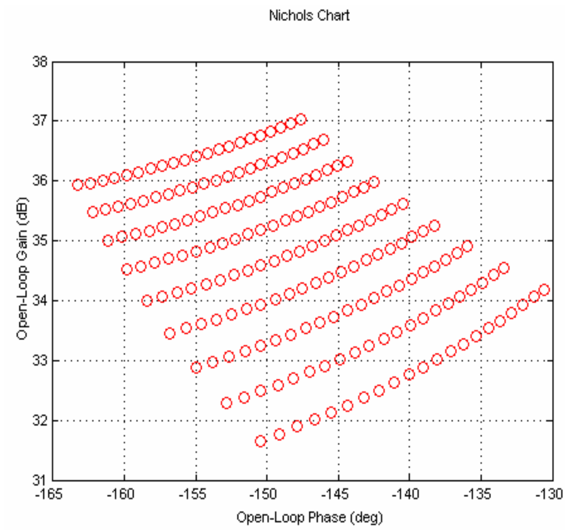

Fig. 11. Template of $(PG_n)_{11}$ at $\omega = 0.1$ rad/s.

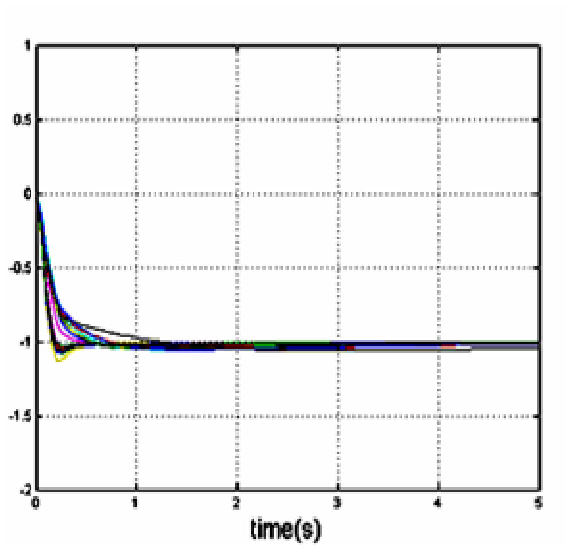

Fig. 9. Time-domain simulation of control signal u_2 due to unit step disturbances.

6. Conclusion

A new and simple method was introduced to design off-diagonal elements of non-diagonal controllers for uncertain MIMO LTI systems by assuming that the controller of the system is the matrix product of the two matrices: one is diagonal and other is non-diagonal. This paper offered a new simple approach to design non-diagonal part of the controller based of the minimization of the template size especially in small frequencies. Then, the design of diagonal part of the controller was done. The delta-norm definition of the matrices was used to quantify the template sizes. The results show the successfully and simplicity of the method. Also a new formulation was developed to the design of non-diagonal robust regulator under certain hard time-domain constraints on output and control signals of the system in response to the step disturbances. The role of off-diagonal elements of the controller was established clearly in the formulation. This theoretical enhancement has not been introduced before. Simulations of the resulting controller show the satisfaction of the specifications but they were somewhat conservative.

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