Bayesian Estimation of Reliability of the Electronic Components Using Censored Data from Weibull Distribution: Different Prior Distributions

Fernando Antonio Moala Débora Luzia Penha

Fernando Antonio Moala Faculty of Sciences and Technology – UNESP 19060-900 - Presidente Prudente - SP - Brazil.

KEYWORDS	ABSTRACT
	The Weibull distribution has been widely used in survival and engineering reliability analysis. In life testing experiments is fairly common practice to terminate the experiment before all the items have failed, that means the data are censored. Thus, the main objective of this paper is to estimate the reliability function of the Weibull distribution with uncensored and censored data by using Bayesian estimation. Usually it is assigned prior distributions for the parameters (shape and scale) of the Weibull distribution. Instead, we assign prior distributions for the reliability function for a fixed time, that is, for the parameter of interest. For this, we propose different non-informative prior distributions for the reliability function and select the one that provides more accurate estimates. Some examples are introduced to illustrate the methodology and mainly to investigate the performance of the prior distributions proposed in the paper. The Bayesian analysis is conducted based on Markov Chain Monte Carlo (MCMC) methods to generate samples from the posterior distributions.

1-Introduction

Life testing is an important statistical method for evaluating the reliability of electronics components and systems. With the advent of new technologies and manufacturing processes, the electronics operate for long time before failing hence the cost of a life test becomes more expensive. In this case, the life tests are usually finished after a specific period of time (Type I censoring) or after a specific number of failures have been observed (Type II censoring). Numerous references for life test planning are available in the statistical and engineering literature, some of them are [1] and some more specific in engineering are given by [2], [3] and [4].

The Weibull distribution is one of the most widely used distribution for analyzing lifetime data, particularly when the data are censored, due to its versatility to model a variety of lifetime data and properties. A detailed discussion about its applications and properties has been given by [5] and [6].

Several methods have been proposed in the literature to estimate the parameters and the reliability of the Weibull distribution, which the Maximum likelihood and Bayesian approaches are the most important.

In many situations, the maximum likelihood estimation (MLE) provides satisfactory estimates mainly as the sample size is large, however in reliability analysis, we can have small and highly censored samples such that the MLE estimation of reliability functions turns out to

* **Corresponding author:** *Fernando Antonio Moala*

Email:femoala@fct.unes.br

be imprecise or even unreliable (see for instance [7]). In this case, Bayesian inference is desirable for estimation of parameters.

The estimation of parameters from Weibull distribution through Bayesian analysis based on Types I and II censored data has been considered by several authors. [8] explore and compare the performance of Maximum Likelihood and Bayesian for estimating the survival function. Some comparisons of estimation methods for Weibull parameters using complete and censored samples have been discussed by [9]. Others include [10], [11].

The use of Bayesian inference to estimate an unknown parameter requires specify a prior distribution for the parameter of interest. Usually it is assigned prior distributions for the parameters (shape and scale) of the Weibull distribution in order to estimate de reliability. Instead, in this paper we propose prior distributions for the reliability function, that is, for the parameter of interest.

Thus, the main goal of this paper consist on proposing different non-informative priors for the reliability function with uncensored and censored data, and select the one that provides more accurate estimates. Bayes point estimates and credible intervals for the reliability are also provided.

Firstly, Jeffreys [12] and reference [13], [14] priors are derived for the reliability in the presence of uncensored data. When the issue contains censored data we consider the proper priors Beta and Negative Log-gamma for the reliability under the assumption of independence of the parameters. However, in some cases we cannot assume independence of the parameters, then we propose an alternative prior distribution based on the copula function (see for example, [15] or [16-17]).

A special case is given by the Farlie-Gumbel-Morgenstern copula [18].

Secondly, we need to choose one appropriate prior distribution for the reliability in each situation: with uncensored and censored data set.

Due to the complex analytical form of the posterior distributions obtained, we have used the Markov chain Monte Carlo (MCMC) techniques as the popular Gibbs sampling algorithm (see for example [19] and [20] or the Metropolis-Hastings algorithm [21] to simulate samples from the joint posterior in order to compute the Bayes estimators and also to construct the posterior distribution for the reliability.

The paper is organized as follows: in Section 2, we derive the prior densities for the reliability by using Jeffrey and reference priors. Sections 3 and 4 show the Bayesian analysis with censored data type II and I, respectively. Three examples consisted on real lifetime data set are introduced in Section 5. Finally, in section 6 we present the conclusions.

2. Estimation of The Reliability for Uncensored Data

In many applications we concentrate our interest on the reliability function *R(t)* of a process, defined as
$$R(t) = P\{X > t\}, \text{ for } x > 0.$$
Let *X* be the lifetime of a component with a Weibull distribution, denoted by W(α, β) and given by

$$f(x \, ; \, \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\} \, , \text{ for all } x \geq 0 \tag{1}$$

thus the reliability function is

$$R(t) = \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right] \tag{2}$$

Let $x_1, \ldots, x_n$ be a random sample of *n* lifetime observations on *X*, then the likelihood function for the parameters α and β, based on *x*, is given by

$$L(\alpha, \beta | x) = \left(\frac{\beta}{\alpha}\right)^n \prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta-1} \exp\left\{-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right\}. \tag{3}$$

After simple but tedious calculations we find that the Fisher information matrix is given by

$$I(\alpha, \beta) = \begin{bmatrix} \left(\dfrac{\beta}{\alpha}\right)^2 & -\dfrac{\psi(2)}{\alpha} \\[3mm] -\dfrac{\psi(2)}{\alpha} & \dfrac{\psi'(2) + \psi(2)^2 + 1}{\beta^2} \end{bmatrix}, \tag{4}$$

where $\psi(2)$ and $\psi'(2)$ are called digamma and trigamma functions, respectively. See [22] for details of these calculations.

To derive the asymptotic properties for the estimator reliability $\hat{R}(t)$ we first transform *(α, β)* to *(R, W)*

where $W = \beta$ and $R = \exp\left\{-\left(\frac{t}{\alpha}\right)^\beta\right\}$. Thus, from (3), the likelihood function in these new parameters (R, W) is given by

$$L(R, W | x) = \frac{1}{W^n}\left(\ln\frac{1}{R}\right)^n \prod_{i=1}^{n} y_i^{W-1} R^{\sum_{i=1}^{n} y_i^w}, \quad (5)$$

where $y_i = \dfrac{x_i}{t}$, $0 < R < 1$, $W > 0$.

[23] derive the expected information matrix for the parameters (R, W), as

$$I(R,W) = \begin{bmatrix} \dfrac{1}{R^2\left(\ln\frac{1}{R}\right)^2} & \dfrac{\psi(2) - \ln\left(\ln\frac{1}{R}\right)}{RW\ln\frac{1}{R}} \\ \dfrac{\psi(2) - \ln\left(\ln\frac{1}{R}\right)}{RW\ln\frac{1}{R}} & \dfrac{1}{W^2}\left\{\ln^2\left(\ln\frac{1}{R}\right) - 2\psi(2)\ln\left(\ln\frac{1}{R}\right) + d\right\} \end{bmatrix}, \quad (6)$$

where $d = \psi'(2) + \psi(2)^2 + 1$.

In a Bayesian framework it is necessary to specify a prior distribution over the parameter space. If prior information is unavailable for a process, then initial uncertainty about the parameters can be quantified with a non-informative prior distribution.

A well known non-informative prior was proposed by [12]. If the variable has density $f(x; \theta)$, where θ is an unknown parameter (scalar or vector), then the Jeffreys prior, denoted by $\pi(\theta)$, is proportional to the square root of the determinant of the Fisher information $I(\theta)$, that is,

$$\pi(\theta) \propto \sqrt{\det I(\theta)}. \quad (7)$$

Jeffreys prior is widely used due to its invariance property under one-to-one transformations of parameters. See [24] for more details.

Thus, from (6) and (7) the Jeffreys prior for the parameters R and W is given by:

$$\pi(R, W) = \frac{1}{RW\ln\frac{1}{R}}. \quad (8)$$

Reference prior is another one that could be applied for the reliability analysis. It was introduced by [13] and further developed by [14].

An important feature in the Berger–Bernardo approach to construct a no informative prior is the different treatment for interest and nuisance parameters. When there are nuisance parameters (typical case in this article), one must establish an ordered parameterization with the parameter of interest singled out and then follow the procedure below.

Suppose that the joint posterior distribution of (ϕ, λ) is asymptotically normal with covariance matrix $S(\hat{\phi}, \hat{\lambda}) = I^{-1}(\hat{\phi}, \hat{\lambda})$. Thus, under appropriate regularity conditions, the joint reference prior can be written as the product of two independent functions of parameters as follows.

Theorem: If the nuisance parameter space $\Lambda(\phi) = \Lambda$ is independent of ϕ, and the functions $S_{11}^{-\frac{1}{2}}(\phi, \lambda)$ and $I_{22}^{\frac{1}{2}}(\phi, \lambda)$ factorize in the form of:

$$S_{11}^{-\frac{1}{2}}(\phi, \lambda) = g_1(\phi)h_1(\lambda), \quad I_{22}^{\frac{1}{2}}(\phi, \lambda) = g_2(\phi)h_2(\lambda) \quad (9)$$

then

$$\pi(\phi) \propto g_1(\phi) \quad . \quad \pi(\lambda|\phi) \propto h_2(\lambda), \quad (10)$$

and the reference prior related to the parametric value ordered (ϕ, λ) is given by:

$$\pi(\phi, \lambda) = g_1(\phi)h_2(\lambda) \quad (11)$$

In this case, there is no need for compact approximation, even if the conditional reference prior is not proper.

Now, consider the construction of reference prior for the reliability function $R(t)$ given in (2).

Our parameter of interest is $R = \exp\left\{-\left(\frac{t}{\alpha}\right)^\beta\right\}$ and the nuisance parameter is chosen to be, for computational convenience, $W = \beta$, but the answer is invariant w.r.t. the choice of W.

The information matrix $I(R, W)$ is given by (6) and its inverse matrix $S(R, W)$ is:

$$S(R,W) = \frac{6}{\pi^2}\begin{bmatrix} R^2\left(\ln\frac{1}{R}\right)^2\left\{\ln^2\left(\ln\frac{1}{R}\right) - 2\psi(2)\ln\left(\ln\frac{1}{R}\right) + d\right\} & -WR\ln\frac{1}{R}\left[\psi(2) - \ln\left(\ln\frac{1}{R}\right)\right] \\ -WR\ln\frac{1}{R}\left[\psi(2) - \ln\left(\ln\frac{1}{R}\right)\right] & W^2 \end{bmatrix}. \quad (12)$$

Therefore, the reference prior for *(R, W)* is derived as follows:

$$S_{11}^{-\frac{1}{2}}(R,W) = \cfrac{1}{R\ln\cfrac{1}{R}\sqrt{\ln^2\left(\ln\cfrac{1}{R}\right)-2\psi(2)\ln\left(\ln\cfrac{1}{R}\right)+d}} = g_1(R)h_1(W),$$

(13)

with $g_1(R) = \cfrac{1}{R\ln\cfrac{1}{R}\sqrt{\ln^2\left(\ln\cfrac{1}{R}\right)-2\psi(2)\ln\left(\ln\cfrac{1}{R}\right)+d}}$, such that

$$\pi(R) = \cfrac{1}{R\ln\cfrac{1}{R}\sqrt{\ln^2\left(\ln\cfrac{1}{R}\right)-2\psi(2)\ln\left(\ln\cfrac{1}{R}\right)+d}} .$$

(14)

Similarly,

$$I_{22}^{\frac{1}{2}}(R,W) = \frac{1}{W}\sqrt{\ln^2\left(\ln\frac{1}{R}\right)-2\psi(2)\ln\left(\ln\frac{1}{R}\right)+d} = g_2(R)h_2(W)$$

(15)

with $h_2(W) = \dfrac{1}{W}$, and thus

$$\pi(W\mid R) = \frac{1}{W}.$$

(16)

Hence, using the theorem before, the joint reference prior with parameter of interest *R* is:

$$\pi(R,W) = \cfrac{1}{WR\ln\cfrac{1}{R}\sqrt{\ln^2\left(\ln\cfrac{1}{R}\right)-2\psi(2)\ln\left(\ln\cfrac{1}{R}\right)+d}}$$

(17)

In Bayesian inference, all the uncertain quantities are modeled in terms of their joint prior distribution and then they are updated, given the data sample, in terms of the joint posterior distribution. Therefore, the joint posterior for the parameters *(R , W)* is given by:

$$p(R,W|x) \propto \frac{1}{W^n}\left(\ln\frac{1}{R}\right)^n \prod_{i=1}^n y_i^{1/W-1} R^{\sum_{i=1}^n y_i^{1/W}} \pi(R,W),$$

(18)

where $\pi(R,W)$ is a appropriate joint prior with $0 < R < 1$ and $W > 0$.

Although the priors (8) and (17) on *R* and *W* are improper the corresponding posteriors are proper's.

Now, integrating (18) with respect to *W* yields the marginal posterior of *R* as

$$p(R\mid x) = c\left(\ln\frac{1}{R}\right)^n \int_0^\infty \frac{1}{W^n}\prod_{i=1}^n y_i^{1/W-1} R^{\sum_{i=1}^n y_i^{1/W}} \pi(R,W)dW,$$

(19)

Where;

$$c^{-1} = \left(\ln\frac{1}{R}\right)^n \int_0^\infty\int_0^\infty \frac{1}{W^n}\prod_{i=1}^n y_i^{1/W-1} R^{\sum_{i=1}^n y_i^{1/W}} \pi(R,W)dWdR,$$

(20)

is the normalizing constant.

Clearly, it is so difficult to find an analytic solution for integrals (19) and (20) and it is necessary to employ software to perform a numerical integration. In this case, we need to appeal to the MCMC (Markov Chain Monte Carlo) algorithm to obtain the marginal posterior density for and summaries of interest.

3. Estimation of the Reliability for Censored Data Type II

In a censored scheme type II, we placed the units on test and terminated the experiment after the occurrence of r-th failure (r fixo, $1 \le r \le n$). This means that the observed data consist of the smallest r observations. The number of failures r is fixed before

the experiment is run. The likelihood function based on r the failures time $X_{(1)}, \cdots, X_{(r)}$ is given by

$$L = \frac{n!}{(n-r)!} f(x_{(1)}) \ldots f(x_{(r)})[S(x_{(r)})]^{n-r}. \tag{21}$$

For lifetimes of the components following the Weibull distribution given in (1), the likelihood

$$L(\alpha, \beta \mid x) = \left(\frac{\beta}{\alpha}\right)^r \prod_{i=1}^{r}\left(\frac{x_{(i)}}{\alpha}\right)^{\beta-1} \exp\left\{-\sum_{i=1}^{r}\left(\frac{x_{(i)}}{\alpha}\right)^{\beta} - (n-r)\left(\frac{x_r}{\alpha}\right)^{\beta}\right\}. \tag{22}$$

For a Bayesian analysis of this censored scheme, we could consider different prior distributions for reliability $R(t)$.

An alternative approach to obtain the posterior distribution for the reliability function could be considered. Instead to use a prior distribution for the parameters α and β of Weibull model we could use a

distribution under range in the interval [0, 1] to represent the prior distribution for the reliability $R(t)$, at time t fixed.

A prior distribution for R that could be used is the Beta distribution, with density given by

$$\pi(R \mid \upsilon, \gamma) = \frac{1}{B(\upsilon, \gamma)} R^{\upsilon-1}(1-R)^{\gamma-1}, 0 \leq R \leq 1, \tag{23}$$

where $\upsilon, \gamma > 0$ and $B(\upsilon, \gamma)$ is the beta function with $B(\upsilon, \gamma) = \frac{\Gamma(\upsilon)\Gamma(\gamma)}{\Gamma(\upsilon+\gamma)}$. The Beta distribution is rich in shapes and has mean and variance given by $\frac{\upsilon}{\upsilon + \gamma}$ and $\frac{\upsilon\gamma}{(\upsilon + \gamma)^2 (\upsilon + \gamma + 1)}$, respectively.

The family of Beta distributions is frequently used as prior model for reliability of a component in binomial and Pascal sampling [25].

The Beta(0, 0) distribution is an improper prior and sometime also is used to represent ignorance of parameter values.

Now, since W is a nonnegative variable in (11), it is assumed that the prior distribution for R is Beta(ϑ, γ) and for W is Gamma(a, b), resulting in the joint prior density given by

$$\pi(R, W) \propto R^{\upsilon-1}(1-R)^{\gamma-1}W^{a-1}\exp\left\{-\frac{W}{b}\right\}, \ 0 \leq R \leq 1, W > 0, \tag{24}$$

with hyper parameters $\vartheta > 0, \gamma > 0, \ a > 0$ and $b > 0$.

For the censored scheme type II, the likelihood function for R and W can be found by substituting

the Weibull parameters α and β in (22) by $R = exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\}$ and $W = \beta$. Thus, the likelihood function may be rewritten as:

$$L(R, W \mid x) = W^r \left(\ln\frac{1}{R}\right)^r \prod_{i=1}^{r} y_i^{W-1} R^{\sum_{i=1}^{r} y_i^{W} - (n-r) y_i^{W}}. \tag{25}$$

Other prior specification for the reliability function consists of assigning a Negative Log-Gamma

distribution for R. In this case, the density with k and λ hyper parameters is given by:

$$\pi(R \mid \lambda, k) = \frac{\lambda}{\Gamma(k)}\left(\ln\frac{1}{R}\right)^{k-1} R^{\lambda-1}, \ 0 \leq R \leq 1, \tag{26}$$

denoted by $NLG(k, \lambda)$. It can be shown that the mean and variance are given by $\left(1 + \frac{1}{\lambda}\right)^{-k}$ and $\left(1 + \frac{2}{\lambda}\right)^{-k} - \left(1 + \frac{1}{\lambda}\right)^{-2k}$, respectively.

It is not difficult to show that the cumulative function for R corresponding to the above equation is given by:

$$F(R \mid \lambda, k) = 1 - I\left(k, \lambda \ln \frac{1}{R}\right), 0 \le R \le 1, \tag{27}$$

where $I(c, d)$ is the incomplete gamma function. [26] provides a discussion of the negative log-gamma distribution.

According to [25], the Negative-Log Gamma distribution represents an alternative procedure in selecting prior distributions when the practitioner is:

$$\pi(R, W) \propto \left(\ln \frac{1}{R}\right)^{k-1} R^{\lambda-1} W^{a-1} \exp\left\{-\frac{W}{b}\right\}, \quad 0 \le R \le 1, W > 0, \tag{28}$$

with hyper parameters $k > 0, \lambda > 0,\ a > 0$ and $b > 0$.

Now, assuming dependence between the parameters R and W, we also could consider a bivariate prior

$$\pi(R, W \mid \rho) = f_1(R) f_2(W) + \rho f_1(R) f_2(W)[1 - 2F_1(R)][1 - 2F_1(W)], \tag{29}$$

where $f_1(R)$ and $f_2(W)$ are the marginal densities for the random quantities R and W; $F_1(R)$ and $F_2(W)$ are the corresponding marginal distribution functions.

more knowledgeable about reliability properties than just the failure rate characteristics of the item under study. Again a gamma distribution would be assigned to the parameter W and joint prior density will be given by:

distribution derived from copula functions (see for example, [15], [16] and [27]).

A special case is given by the Farlie-Gumbel-Morgenstern Copula [18] with corresponding bivariate prior distribution for R and W given ρ, is given by

Observe that if $\rho = 0$, we have independence between R and W.

In this article, we will assume Gamma marginal distribution for W and both joint Beta and Negative Log-Gamma marginal distributions for the reliability R, resulting in two different prior distributions for (R, W). They are respectively given by:

$$\pi_1(R, W \mid \rho) \propto R^{\upsilon-1}(1-R)^{\gamma-1} W^{a-1} \exp\left\{-\frac{W}{b}\right\}[1 + \rho[1 - 2I_R(\upsilon, \gamma)][1 - 2I(a, W/b)]], \tag{30}$$

and

$$\pi_2(R, W \mid \rho) \propto \left(\ln \frac{1}{R}\right)^{k-1} R^{\lambda-1} W^{a-1} \exp\left\{-\frac{W}{b}\right\}\left[1 + \rho[I(k, \lambda \ln \frac{1}{R}) - 1][1 - 2I(a, W/b)]\right], \tag{31}$$

with $0 < R < 1$ and $W > 0$. $I_x(u, z)$ is the cumulative function of the Beta variable and it is called Incomplete Beta function.

The posterior distributions for the parameter (R, W, ρ) are obtained multiplying the likelihood function (25) and each prior distribution in (30) and (31), that is,

$$p_i(R, W \mid x) \propto L(R, W \mid x)\pi_i(R, W \mid \rho)\pi(\rho), i = 1, 2,$$

where $\pi(\rho)$ is a prior distribution for ρ.

In general, many different priors can be used for ρ; one possibility is to consider

$$\pi(\rho) \propto (1 - \rho^2)^c, \tag{32}$$

with a specified value for the constant c ; other possibility is to consider a uniform prior distribution for ρ over the interval $[-1, 1]$.

4. Estimation of the Reliability for Censored Data Type I

In a censored scheme Type I, the test is terminated when a pre-specified time point, L, on test has been reached. One more complicated form of type I censored data is to consider an experiment where each tested component has its specific censored time, because not all units start the test on the same date. A Type-I censoring sampling scheme can be described as follows.

Suppose n electronic components are placed on a life testing experiment. The lifetimes of the sample units are independent and identically distributed (i.i.d.) random variables. It is assumed that the failed items are not replaced.

We consider $L_1, \cdots, L_n$ the censored times are fixed for each unit in the experiment. The lifetime X_i of the ith unit will be observed if $X_i \le L_i, i = 1, \cdots, n$. Therefore, the observed data is (t_i, δ_i), where

$$t_i = min(X_i, L_i), \tag{33}$$

and

$$\delta_i = \begin{cases} 1 \ if \ T_i \leq L_i \\ 0 \ if \ T_i > L_i \end{cases}, \qquad (34)$$

is the indicator random variable δ_i which implies if T_i is censored or not.

where $S(t)$ is the survival function.

In this censored scheme, the likelihood function on the parameters (R, W) is given by

where
$$T^* = \sum_{i=1}^n \delta_i y_i^{\hat{\beta}} + \sum_{i=1}^n (1 - \delta_i)(L_i / t)^{\hat{\beta}} \text{ and}$$

$y_i = \dfrac{x_i}{t}$.

For a Bayesian analysis of the reliability under type I censored data we consider the same prior distributions discussed in the previous section, but the likelihood will be given by equation (36).

5. Numerical Illustration

In this section we introduce three practical examples with real lifetime data set to illustrate the proposed approaches and for comparing the performance of the prior distributions presented in this article.

The maximum likelihood and Bayesian estimators are also compared.

probability given by the Metropolis ratio

$$\alpha(R_i, R_{i+1}) = min \left\{ 1, \frac{B\left(\frac{bR_i}{1-R_i}, b\right) p(R_{i+1}, W_i \mid t)}{B\left(\frac{bR_{i+1}}{1-R_{i+1}}, b\right) p(R_i, W_i \mid t)} \right\}; \qquad (37)$$

iv) sample the new value W_{i+1} from the Gamma distribution $G(W_i / d, d)$;

v) the candidate W_{i+1} will be accepted with a probability given by the Metropolis ratio

$$\alpha(W_i, W_{i+1})$$
$$= min \left\{ 1, \frac{G(W_i / d; d) p(R_{i+1}, W_{i+1} \mid t)}{G(W_{i+1} / d; d) p(R_{i+1}, W_i \mid t)} \right\} \qquad (38)$$

The proposal distribution parameters and were chosen to obtain good mixing of the chains and the chain is run for 25000 iterations with a burn-in period of 5000. The convergence of the MCMC samples of parameters is assessed using the criteria of Raftery and Lewis [28] diagnostic.

5-1. Fist Example (Uncensored Data)
This example was proposed by [29] on the fatigue life of ball bearings. The following observations denote the number of cycles to failure:
17.88, 28.92, 33, 41.52,42.12,45.6,48.48, 51.84, 51.96, 54.12, 55.56, 67.80,

If the pairs (t_i, δ_i) are independent, then the likelihood function is given by:

$$L = \prod_{i=1}^n \left[f(t_i) \right]^{\delta_i} \left[S(L_i) \right]^{1-\delta_i} \qquad (35)$$

,

$$L(R, W \mid x) = W^r \left(\ln \frac{1}{R} \right)^r \left(\prod_{i=1}^r y_i^{(W-1)\delta_i} \right) R^{T^*} \quad (36)$$

As the joint posteriors seem quite intractable and therefore we cannot obtain marginal posteriors analytically we need to employ Markov chain Monte Carlo (MCMC) techniques to determine them.

Specifically, we run an algorithm for simulating a long chain of draws from the posterior distribution, and base inferences on posterior summaries of the parameters or functional of the parameters calculated from the samples. Details of the implementation of the MCMC algorithm used in this paper are given below:

i) choose starting values R_0 and W_0 with $0 < R_0 < 1$;

ii) At step $i+1$, we draw a new value R_{i+1} conditional on the current R_i from the Beta distribution $B(bR_i / (1 - R_i, b)$;

iii) the candidate R_{i+1} will be accepted with a

68.64, 68.88, 84.12,93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

[29] argued that the Weibull is a suitable model for describing these data.

The maximum likelihood estimator of reliability for $t = 100$ is approximately 0.22.

For a Bayesian analysis of the data let us assume the prior distribution (8) and (17) for the vector (R, W). Using the software R we first simulated 5,000 MCMC samples ("burn-in-samples") for the joint posterior distribution for R and W that were discarded to eliminate the effect of the initial values used in the iterative simulation method.

After this "burn-in-period", we simulated other 20,000 samples. The convergence of the Gibbs sampling algorithm was monitored from trace plots of the simulated samples.

The posterior summaries of interest and credible intervals considering the Jeffreys and Reference prior distributions are given in Table 1.

Tab.1. Posterior summaries and interval for R(100)from Prior

Estimators	Jeffrey's	Reference
Mean	0.239	0.244
Variance	0.005	0.005
Credible interval	(0.116; 0.389)	(0.119; 0.404)

According to the results shown in Table 1, the probability of the lifetime of ball bearings is greater than 100 cycles is 0.24. It is also observed that the Jeffreys and reference priors adopted for solving this problem have produced similar results.

For comparison of the both priors proposed in this example, the marginal posteriors resulting for R are plotted in Figure 1.
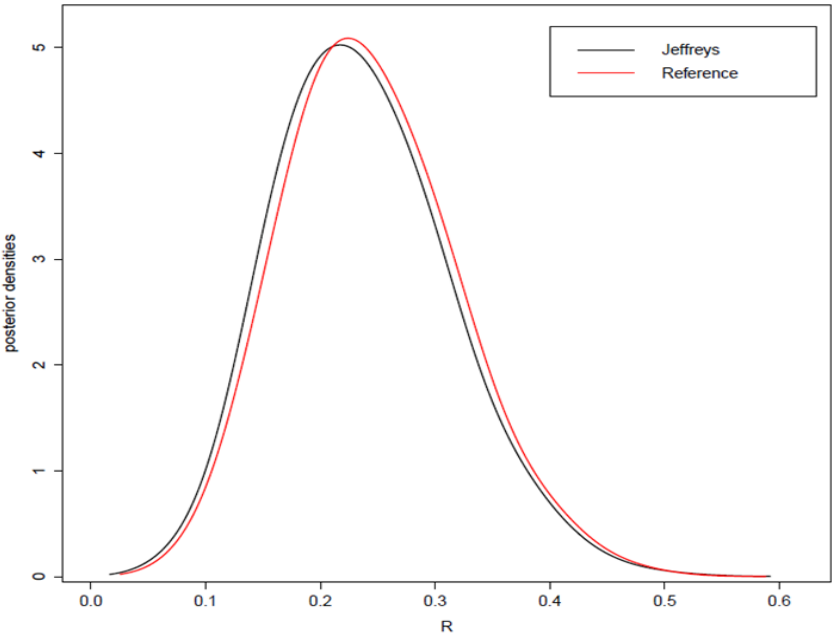

Fig.1. Plots of marginal posterior distributions of reliability at $t = 100$

5-2. Second Example (Presence of Censored Data Type I)

[30] considers a situation in which pieces of equipment are installed at different times. At a later date some of

the pieces will have failed and the rest will still be in use. The data is arranged in Table 2, showing results for 10 pieces of equipment. The life test in question was terminated on August 31.

Tab.2. Operating Times for 10 Pieces of Equipment

Item Number	1	2	3	4	5
Date of installation	11 June	21 June	22 June	2 July	21 July
Date of failure	13 June	---	12 August	----	23 August
Lifetime (days)	2	≥ 72	51	≥ 60	33
Item Number	6	7	8	9	10
Date of installation	31 July	31 July	1 August	2 August	10 August
Date of failure	27 August	14 August	25 August	6 August	---
Lifetime (days)	27	14	4	4	≥ 21

At that time three items (numbers 2, 4, and 10) had still not failed, and their failure times are therefore

censored; we know for these items only that their failure times exceed 72, 60, and 21 days, respectively.

Item	1	2	3	4	5	6	7	8	9	10
T_i	2	--	51	--	33	27	14	24	4	--
L_i	81	72	70	60	41	31	31	30	29	21

The lifetime of items 2, 4 and 10 are censored. It will be noted that the effective censoring times are known for all items. The data give $r = 7$.

Now we compute the Bayes estimators of R for $t = 30$. Assuming there is no prior information we consider the following prior distributions

i) $R \sim Beta(0,0)$ and $W \sim \Gamma(0.01, 0.01)$.
i) $R \sim LGN(0.1, 0.01)$ and $W \sim \Gamma(0.01, 0.01)$.
In this example we also consider the copula prior (29) with log-gamma negative and gamma marginal distributions for R and W, respectively. We will assume known the hyper parameters with $a_1 = b_1 = a_2 = b_2 = 0.01$ to have noni formative prior.

We also need to appeal to numerical procedures to extract characteristics of marginal posterior distributions such as Bayes estimator and credible intervals.
We can then use MCMC algorithm.
The resulting Bayesian estimators for the reliability is given in Table 3 and the marginal posterior densities under the three priors are displayed in Figure 2.

Tab.3. Posterior summaries and interval for R(30) from each Prior

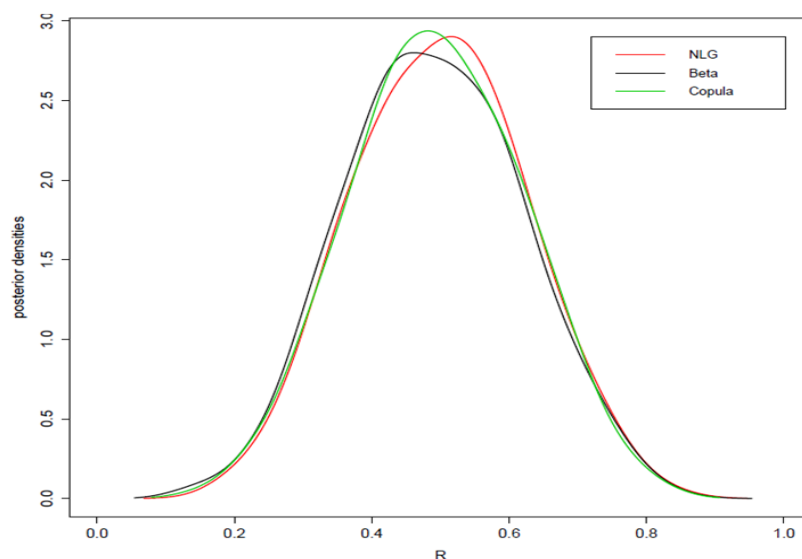| Estimators	Beta	LGN	Copula
Mean	0.491	0.499	0.494
Variance	0.016	0.016	0.015
Credible interval	(0.246; 0.740)	(0.261; 0.742)	(0.249; 0.739)

Fig.2. Marginal posterior densities for the parameter R with $t = 30$

Comparison of the point estimators provided by the priors shows there is no difference among them and similar result is observed among the three credible intervals. Examining the Figure 2 we observe that the three classes of priors provides quite similar posterior densities.
[1] also discusses this example but with an exponential distribution fitted to this dataset. The maximum likelihood estimator for the reliability parameter is 0.506 and the 95% confidence interval is [0.07; 0.676]. Note that the Bayesian intervals are smaller than this one.

5-3.Third Example (Presence of Censored Data Type II)
[31] provide data on the failure times of aircraft components subject to a life test. The data are obtained

from $n = 13$ randomly selected test items and the life test terminate at the observed failure of the 10th item.
The $r = 10$ observed lifetimes were (in hours):
0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.34. 1.76. 2.50, 3.00.
These data have been discussed by [1], who fitted a Weibull model to these data. The maximum likelihood estimator for the reliability at $t = 2$ (for example) is given by $\hat{R} = 0.434$. Thus, the estimated probability of the lifetime of components of an aircraft is greater than two hours is approximately 0.22.
For a Bayesian analysis of the data let us assume the same prior distributions proposed in the second example. The Bayesian summaries, obtained numerically by the MCMC algorithm, are displayed in Table 4.

Tab.4. Posterior summaries and credible interval for $R(2)$ from each prior.

Estimators	Beta	LGN	Copula
Mean	0.212	0.224	0.222
Variance	0.010	0.011	0.011
Credible interval	(0.056; 0.444)	(0.066; 0.465)	(0.064; 0.457)

From the results of Table 4, we observe similar results assuming the priors studied in this article. However, the maximum likelihood estimator for reliability differs considerably from the Bayesian estimator.

We also plot in Figure 3 the posterior densities from each prior distribution studied. Observe that we have in this example a small size (n = 13 observations).
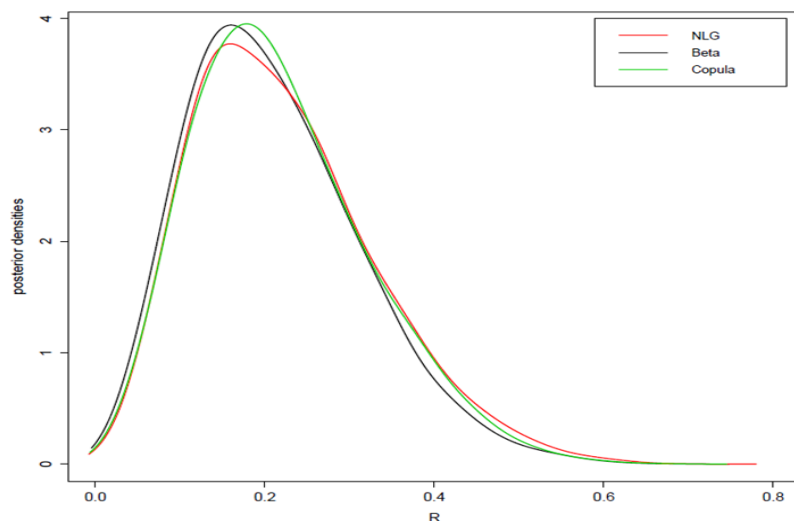

Fig.3. Densidades marginais a posteriori para o parâmetro R.

6. Concluding Remarks

In this article, we analyze different non-informative prior distributions for the estimation of reliability applied to the Weibull model under censored and uncensored data. The performance of the priors is examined by comparing their posterior summaries and intervals. The motivation for this work is particularly due to the importance of using non-informative priors when relatively little prior information is available and mainly due to the informative priors, based on subjective methods, are usually more difficult to apply in practice situations.

We have explored two important classes of non-informative prior distributions to estimate $R(t)$ for uncensored data. The focus is on the comparison of Jeffreys and reference priors and both priors seem to perform equally for the Example 1. However, if we had to choose one of them, the Jeffreys prior would be recommended since it is easier to use and due to having the property of being invariant under transformations while the reference prior is typically difficult to use in practice. Moreover, it has undesirable properties including lack of invariance to parameterization and no uniqueness of prior due to the choice of the parameter of interest.

In the case of censored data it is not always possible to obtain the expected information matrix and consequently the Jeffreys and Reference priors can not be obtained.

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In the presence of censored data we firstly consider the Bayesian analysis by assuming independence of parameters R and W with Beta and Negative log-gamma distributions as priors for the reliability $R(t)$, which has not often been considered in the literature as priors for this parameter.

Assuming dependence between the parameters we also propose a bivariate prior distribution derived from the Farlie-Gumbel-Morgenstern copula. The copula approach could be a good alternative to provide a noni formative prior for the parameters. Its flexibility and analytical tractability suggest that it is a promising way to represent dependence. Furthermore, side conditions reflecting initial information about the marginal distributions for parameters that may be available can readily be utilized in deriving copula prior.

Although the results show that priors had the same performance for the Examples 2 and 3, we recommend the copula prior distribution for routine applied use.

Our study has focused on the Weibull distribution under type I and II censored data however, it would be of interest to conduct similar studies for the progressive censoring scheme. Another important research we are also interested consist on propose a subjective prior distribution based on the expert information in order to improve the accuracy of the parameter estimation.

References

[1]   Lawless, J.F.,*" Statistical Models and Methods for Lifetime Data"*. Wiley & Sons. New York (2011).

[2]   O'Connor, P.; Kleyner, *A Practical Reliability Engineering*. Wiley & Sons. New York (2012).

[3]   Karbasian, M.; Mehr, M.R.; Agharajabi, M.,"*Improvement of the Reliability of Automatic Manufacture Systems by Using FTA Technique"*. International Journal of Industrial Engineering & Production Research. Vol. 23, No. 2, pp 85-89 (2012).

[4]   Naini, S.G.J.; Aryanezhad, M.B.; Jabbarzadeh, A.; Babaei, H.,. *"Condition Based Maintenance for Two-Component Systems with Reliability and Cost Considerations"*. International Journal of Industrial Engineering & Production Research. Vol. 20, No. 3, pp 107-116 (2009).

[5]   Balakrishnan, N.; Basu, P., *"Weibull Distribution: Theory, Methods and Applications"*. Gordon & Breach Publishing Group (1997).

[6]   McCool, J.I., *"Using the Weibull Distribution"*: Reliability, Modeling and Inference. Wiley & Sons. New York (2012).

[7]   Bacha, M., Celeux, G.,. *"Bayesian Estimation of a Weibull distribution in a highly censored and small sample setting"*. INRIA. Rapport de recherché n. 2993 (1996).

[8]   Ahmed, O.M.; Ibrahim, N.A.,. *"Bayesian Survival Estimator for Weibull Distribution with Censored Data"*. Journal of Applied Sciences, 11: 393-396 (2011).

[9]   Hossain, A.M.; Zimmer, W.J.;. *"Comparison of estimation methods for Weibull parameters: complete and censored samples"*. J. Statist. Comput. Simulation 73 (2), 145-153 (2003).

[10]   Zhang, Y., Meeker, W. Q.,. *"Bayesian life test planning for the Weibull distribution with given shape parameter"*. Metrika 61, 237-249 (2005).

[11]   Joarder, A.; Krishna, H.; Kundu, D. *"Inferences on Weibull parameters with conventional type-I censoring"*. Computational Statistics and Data Analysis 55, 111(2011).

[12]   Jeffreys, Sir Harold *"Theory of probability"*. Oxford University Press (1967).

[13]   Bernardo, J. M., *"Reference posterior distributions for Bayesian inference "*(with discussion). J. Roy. Statist. Soc. Ser. B 41 113-147. [Reprinted in Bayesian Inference (N. G. Polson and G. C. Tiao, eds.) 229-263. Edward Elgar, Brookfield, VT, 1995 (1979).

[14]   Berger, J. O., Bernardo, J. M., *"On the development of reference priors"* (with discussion). In Bayesian Statistics 4 (J. M. Bernardo, J.O., Berger, A.P., Dawid, A.F.M., Smith, eds.) 35-60. Oxford Univ. Press (1992).

[15]   Nelsen, R.B., *"An introduction to copulas, Springer-Verlag"*. New York (1999).

[16]   Trivedi, P.K.; Zimmer, D.M., (2005a) *"Copula Modeling, Now Publishers"*.

[17]   Trivedi, P.K.; Zimmer, D.M., (2005b) *"Copula Modelling: an introduction to practitioners"*. Form dations and trends in econometrics, 1, 1, 1-111.

[18]   Morgenstern, D., *"Einfache Beispiele Zweidimensionaler Verteilungen"*. Mitteilingsblatt fur Matematische Statistik, 8, 234-253 (1956).

[19]   Gelfand, A.E.; Smith, F.M., *"Sampling-based approaches to calculating marginal densities"*. Journal of the American Statistical Association, n. 85, pp 398-409 (1990).

[20]   Casella, G.; George, E.I.,"*Explaining the Gibbs sampler"*. The American Statistician, 46, 3, 167, 174 (1992).

[21]   Chib, S.; Greenberg, E., *"Understanding the Metropolis-Hastings algorithm"* The American Statistician, 49,pp 327 – 335 (1995).

[22]   Sun, D., (1997). *"A note on no informative priors for Weibull distributions"*. J. Statist. Planning and Inference 61, pp 319-338.

[23]   Moala, F.A.; Rodrigues, J.; Tomazella, V.L.D., *"A note on the Prior Distributions of Weibul parameters for the Reliability Function"*. Communications in Statistics: Theory and Methods, v. 38, pp 1041-1054 (2009).

[24]   Box, G.E.P., Tiao, G.C., "*Bayesian inference in statistical analysis*". Addison Weiley (1973).

[25]   Martz, H.F., Waller, R.A., "*Bayesian Reliability Analysis*". Wiley & Sons. New York (1982).

[26]   Locks, M.0., "*Reliability, Maintainability and Availability Assessment*", Hayden Books, Rochelle Park, NJ (1973).

[27]   Achcar, J.A.; Moala, F.A., "*Use of MCMC methods to obtain Bayesian inferences for the Birnbaum-Saunders distribution in the presence of censored data and covariates*". Advances and Applications in Statistics, v. 17, pp 1-27 (2010).

[28]   Raftery, A.E., Lewis, S.M., "*One long run with diagnostics: Implementation strategies for Markov chain Monte Carlo*". Statistical Science, 7, 493-497 (1992).

[29]   Leiblein, J., Zelen, M., "*Statistical investigation of the fatigue life of deep groove ball beatings*". J. of Res. Nat'l-burstad. 57, 273-315 (1956).

[30]   Bartholomew, Do J.," *A problem in life testingo* J. Amer. Stat. Assoc. 52: 350-355 (1957).

[31]   Mann, N. R., Fertig, K.W., "*Tables for Obtaining Weibull Confidence Bounds and Tolerance Bound, Based on Best Linear Invariant Estimates of -the Extreme-Value Distribution*," Techno metrics 1.5, 87-101 (1973).