



# An Optimal NPV Project Scheduling with Fixed Work Content and Payment on Milestones

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## KEYWORDS

Project scheduling;  
Net present value

## ABSTRACT

*We consider a project scheduling problem with permitted tardiness and discrete time/resource trade-offs under maximum net present value objective. In this problem, a project consists of a set of sequential phases such that each phase contains one or more sub-projects including activities interrelated by finish-start-type precedence relations with a time lag of zero, which require one or more renewable resources. There is also a set of unconstrained renewable resources. For each activity, instead of a fixed duration and known resource requirements, a total work content respect to each renewable resource is given which essentially indicates how much work has to be performed on it. This work content can be performed in different modes, i.e. with different durations and resource requirements as long as the required work content is met. Based on the cost of resources units and resource requirements of each activity, there is a corresponding cash flow for the activity. Each phase is ended with a milestone that corresponds to the phase income. We prove that the mode corresponding to the minimum possible duration of each activity is the optimal mode in this problem. We also present a simple optima scheduling procedure to determine the finish time of each activity.*

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## 1. Introduction

In this research, we deal with a project scheduling problem under maximum net present value objective in which discrete time/resource trade-offs and tardiness is permitted. We also assume that there is not any constraint on the required renewable resources. This problem considers a contractor that aims to maximize his benefits. The contractor has to implement the project, represented as the activity-on-node (AoN) format, subject to the precedence relations. In order to perform the project activities, some renewable

resources are required that the contractor should hire them for the required time intervals. Thus, performing of each activity involves a series of cash flow payments throughout the activity duration that can be calculated by compounding the associated cash flows to the end of the activity. We also assume each activity has a prescribed work content (man/day or machine/day or ...) with respect to each renewable resource and can be performed in different modes, i.e. with different durations and resource requirements, as long as the required work content is met. The contractor receives his incomes at the end of each phase, considered as the milestones. In addition of the precedence constraints, the project has to be finished before an agreed due date but the tardiness is permitted by paying penalty for each time instant delay.

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Figure 1 depicts the general structure of the project we consider in our research. In this structure, a project consists of a set of interconnected sequential phases such that each phase contains one or more sub-project including a set of activities interrelated by finish-start type precedence relations with time lag of zero. All activities of a project phase except the last one

(milestone) have negative cash flows considered as the project costs. The milestone of each phase has positive cash flow considered as the phase income. The dummy activity zero with zero cash flow is considered as the project start and is begun at time instant zero.

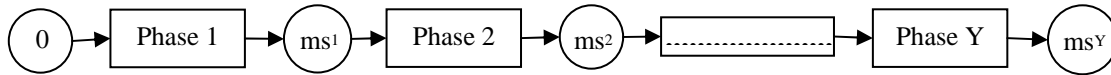


Fig. 1. General structure of the project

The most important application of the proposed problem is in the pharmaceutical industries. The pharmaceutical drug-development process is monitored by the Food and Drug Administration (FDA), and typically follows four main stages: basic research, preclinical, clinical and FDA review, with the clinical stage subdivided in Phase I, II and III. Each sub-stage contains a number of tasks interrelated as a subproject. In this paper we present a solution procedure that result in optimal scheduling without any computational work. We first prove that the mode corresponding to the minimum possible duration of each activity is the optimal mode in this problem. Then, we present an optimal scheduling procedure for the problem with the fixed modes. The remainder of this paper is organized as follows. A review of the literature is given in section 2. Section 3 describes the optimal mode assignment. In section 4, the optimal scheduling procedure for the problem with the fixed modes is presented. Finally, section 5 is reserved for overall conclusions and suggestions for future research.

## 2. Review of the Literature

To the best of our knowledge, the literature on the problem considered in this research is relatively void. Research efforts have been concentrated on some related problems.

The first related problem is the deterministic max-npv problem, denoted as  $cpm, \delta_n, cj/npv$  using the classification scheme of Demeulemeester (2002) that considers the maximization of the project net present value (NPV) subject to the temporal precedence relations. Grinold (1972) showed this problem can be transformed into a linear programming problem and Vanhoucke et al (2001) developed an efficient recursive procedure to find the optimal solution, consisting of three basic steps: the construction of an early tree, the construction of a current tree and a recursive search on the current tree. In the same article, Vanhoucke developed a branch-and-bound procedure based on the mode delaying alternatives and his recursive algorithm for the case of constrained resources denoted as  $m, l|cpm, \delta_n, cj/npv$ . Furthermore, Schwindt and Zimmermann (2001) developed a

steepest ascent algorithm for maximizing the net present value in project networks subject to generalized precedence relations with both minimal and maximal time-lags denoted as  $gpr, \delta_n, cj/npv$ .

The second related problem is the discrete time/resource trade-offs problem (DTRTP) that goal to minimize the project makespan under constraint of a single renewable resource. For the DTRTP, some heuristic solution procedures based on the local searches and tabu search were developed by De Reyck et al. (1998). Demeulemeester et al. (2000) developed a branch and bound procedure based on the activity-mode combination idea to solve DTRTP optimally. Also, Ranjbar et al. (2009) developed an efficient metaheuristic procedure based on the scatter search algorithm for the DTRTP.

The third and the last related problem is the discrete time/cost trade-off problem (DTCTP) in which the duration of each activity is a discrete non-increasing function of the amount of a single nonrenewable resource committed to it (De et al. 1995). The DTCTP is studied under three different objectives: the minimization of the project duration under fixed resource availability  $(1, T|cpm, disc, mu|C_{max})$ , the minimization of the total resource consumption to achieve a target project completion time  $(1, T|cpm, \delta_n, disc, mu|av)$ , and the construction of the efficient time/resource profile over the feasible project durations  $(1, T|cpm, disc, mu|curve)$ . The DTCTP is known to be strongly NP-hard (De et al. 1997); exact algorithms have been presented by Demeulemeester et al. (1996) and Demeulemeester et al. (1998). In addition, Vanhoucke and Debels (2007) consider extension of the DTCTP and develop a few heuristic procedures for them.

## 3. The Optimal Mode Assignment

The objective of the problem defined in this paper is to schedule each activity of a project in one of its defined modes subject to the precedence constraints in order to maximize the net present value of the project. The duration of each activity is not predetermined, but changes as a discrete non-increasing function of the amount of renewable resources assigned to it. The parameters used are defined in table 1.

Tab. 1. Notations

$A = \{0, 1, \dots, n\}$	Set of activities with index $i$
$Y$	Number of project phases with index $\varphi$
$MS = \{ms^1, \dots, ms^Y\}$	Set of milestones
$E = \{(i, j); i, j \in A \cup MS\}$	Set of precedence relations
$P_i$	Set of immediate predecessors of activity $i$
$Q_i$	Set of immediate successors of activity $i$
$R$	Number of renewable resources with index $k$
$N^\varphi = A^\varphi \cup ms^\varphi$	A partial network including activities and corresponding milestone that constitute the phase $\varphi$ ; $\varphi = 1, \dots, Y$
$s_i$	Start time of activity $i$ ; $i \in A$
$f_i$	Finish time of activity $i$ ; $i \in A$
$s^\varphi = \min_{i \in N^\varphi} \{s_i\}$	Start time of phase $\varphi$ ; $\varphi = 1, \dots, Y$
$f^\varphi = \max_{i \in N^\varphi} \{f_i\}$	Finish time of phase $\varphi$ ; $\varphi = 1, \dots, Y$
$w_{ik}$	Work content of activity $i$ with respect to renewable resource $k$ ; $k = 1, \dots, R$
$M_i$	Number of possible modes for activity $i$ ; $i \in A$
$d_{im}$	Duration of activity $i$ in mode $m$ ; $i \in A$ ; $m = 1, \dots, M_i$
$d_i^{\min}$	Minimum possible duration of activity $i$ ; $i \in A$
$r_{imk} = \left\lceil \frac{w_{ik}}{d_{im}} \right\rceil$	resource requirement of activity $i$ in mode $m$ from renewable resource $k$ ; $i \in A$ , $m = 1, \dots, M_i$ , $k = 1, \dots, R$
$a_k$	Cost of one unit of renewable resource $k$ for a time unit; $k \in R$
$g_{imt} = \sum_{k=1}^R r_{imk} a_k$	Cash flow of activity $i$ in mode $m$ and in period $t$ ; $i \in A$ , $t = s_i + 2, \dots, s_i + d_{im}$
	and $g_{im(s_i+1)} = \sum_{k=1}^R (w_{ik} - (d_{im} - 1)r_{imk})a_k$
$dr$	Discount rate
$e^{-\alpha} = (1 + dr)^{-1}$	Discount factor
$c_{im} = \sum_{t=1}^{d_{im}} g_{imt} e^{\alpha(d_{im}-t)}$	Compound associated cash flow occurring at the end of activity $i$ in mode $m$ ; $i \in A$ and $m = 1, \dots, M_i$
$b^\varphi$	Income of phase $\varphi$ ; $\varphi = 1, \dots, Y$

$D$	Project deadline
$P$	Penalty cost for a time unit
$ef_i(ef^\varphi)$	Earliest finish time of activity $i$ (phase $\varphi$ ); $i \in A$ ( $\varphi = 1, \dots, Y$ )
$lf_i(lf^\varphi)$	Latest finish time of activity $i$ (phase $\varphi$ ); $i \in A$ ( $\varphi = 1, \dots, Y$ )
$P_i^\varphi$	The longest path length between activity $i$ and $ms^\varphi$

The project milestones can be considered as dummy activities in terms of having zero duration but for each milestone  $\varphi$ , there is a corresponding income denoted as  $b^\varphi$  and determined by negotiation. All modes of each real activity have to be efficient. A mode is called efficient if every other mode has either a higher duration or a higher resource requirement for at least one resource  $k$ .

When activity  $i \in A$  is performed in mode  $m$ , we assume that it requires  $\left\lceil \frac{w_{ik}}{d_{im}} \right\rceil$  of resource  $k \in R$  in each period  $t = s_i + 2, \dots, s_i + d_{im}$  and requires  $w_{ik} - (d_{im} - 1) \left\lceil \frac{w_{ik}}{d_{im}} \right\rceil$  of resource  $k \in R$  in period  $t = s_i + 1$ . We number modes of each activity based on increasing order of the corresponding durations. Before determining the optimal mode for each activity, we consider the following property.

**Property 1:** The compound cash flow of each activity is an increasing function of its duration.

**Proof:** Based on the mode definition and distribution of resource units over duration of activities, when duration of an activity is increased, its resource requirement is decreased. As the work content in both cases is constant, it means that the resource usage is delayed when activity duration is increased. The delayed resource units are compounded with the positive discount factor  $e^\alpha$  and therefore, the compound cash flow is increased.

Therefore, for a single activity, the mode corresponding to the minimum possible duration is the optimal mode.

In continue, we prove that in a network with structure of figure 1, the minimum possible duration of each activity determines its optimal mode.

**Theorem 1:** The mode corresponding to the minimum possible duration of each activity is the optimal mode.

**Proof:** Consider two duration vectors  $\mathbf{d} = (d_1, \dots, d_j, \dots, d_n)$  and  $\mathbf{d}' = (d'_1, \dots, d'_j, \dots, d'_n)$  in which  $d_i = d'_i$  for  $\forall i \in A \setminus j$  and  $d'_j > d_j$  in which  $j$  is an activity of phase  $\varphi^*$ .

If activity  $j$  is not a critical activity based on the CPM calculations, then  $f_i = f'_i$  for  $\forall i \in A$ . Based on the property 1, we know  $c'_{j1} > c_{j1}$  and thus  $NPV > NPV'$ .

But if activity  $j$  is a critical activity, let  $\Delta = f'_i - f_i = d'_j - d_j$  for  $\forall i \in j \cup Q_j^T$  in which  $Q_j^T$  represents the set of all direct and indirect successors of activity  $j$ .

To compare these two duration vectors, change  $f_i$  as  $f_i = f_i + \Delta$  for  $\forall i \in A^\varphi$  where  $\varphi \geq \varphi^*$ . Now,  $\forall \varphi, f^\varphi = f'^\varphi$  and  $\forall i \in A^\varphi$  where  $\varphi > \varphi^* : f_i = f'_i$  but for each activity  $i \in A^{\varphi^*} \setminus (j \cup Q_j^T)$ ,  $f'_i < f_i$  and also  $c'_{j1} > c_{j1}$ ; therefore,  $NPV > NPV'$ .

#### 4. The Optimal Scheduling Procedure

In previous section, we determined the optimal mode for each activity that corresponds with the minimum possible duration of the activity. Therefore, in this section we consider the defined problem but with fixed modes furthered referred to as PSPNPV (Project Scheduling Problem with Net Present Value objective). We ignore the index of modes ( $m$ ) for notations defined in table 1 to use in this section. Using this modification, the PSPNPV can be formulated as follows:

$$\text{Max } NPV = \sum_{\varphi=1}^Y b^\varphi e^{-\alpha f^\varphi} - \sum_{i \in A} c_i e^{-\alpha f_i} - \max(0, f^\varphi - D) p e^{-\alpha f^\varphi} \quad (1)$$

$$f_i \leq f_j - d_j ; \forall (i, j) \in E \quad (2)$$

$$f_0 = 0 \quad (3)$$

$$f_i \in \mathbf{N} \quad i = 1, 2, \dots, n \quad (4)$$

The nonlinear objective function [1] maximizes the net present value of the project. Constraint [2] indicates the precedence constraints in which duration of milestones is considered zero. Constraint [3] considers  $t = 0$  as the project start time. Finally, constraints [4] ensure that the decision variables are positive integer variables.

In order to find optimal finish time of each activity, we use the following lemmas and definitions.

**Definition 1:** The discounted cash flow of each phase  $\varphi$  ( $DCF^\varphi$ ) is defined as

$$DCF^\varphi = b^\varphi e^{-\alpha(f^\varphi - s^\varphi)} - \sum_{i \in A^\varphi} c_i e^{-\alpha(f_i - s^\varphi)}$$

**Definition 2:** The cumulative discounted cash flow of phases  $\varphi_1$  to  $\varphi_2$  ( $CDCF_{\varphi_1}^{\varphi_2}$ ) is defined as

$$CDCF_{\varphi_1}^{\varphi_2} = \sum_{\varphi=\varphi_1}^{\varphi_2} b^\varphi e^{-\alpha(f^\varphi - s^{\varphi_1})} - \sum_{i \in \bigcup_{\varphi=\varphi_1}^{\varphi_2} A^\varphi} c_i e^{-\alpha(f_i - s^{\varphi_1})}.$$

**Lemma 1:** For each project with structure of figure 1 and including one phase, if  $CCF^1$  is positive,  $f^1 = ef^1$  and  $\forall i \in A: f_i = lf_i$ .

**Proof:** For every particular integer  $f^1$ , it is obvious that all activities  $i \in A$  should be shifted to the right as many as possible ( $\forall i \in A: f_i = lf_i$ ) to maximize  $NPV$ . Therefore, for each activity  $i \in A$ ,  $f_i$  can be shown as  $f_i = f^1 - P_i^1$  where  $P_i^1$  can be easily obtained using usual CPM calculations. On the other hand, we know that  $NPV = DCF^1 \cdot e^{-\alpha s^1}$ . Thus, we obtain the maximum of the  $NPV$ , if we consider the minimum value for  $s^1$  that is zero. In other words,  $f^1$  should take its minimum value ( $f^1 = ef^1$ ) because  $f^1 = s^1 + P_0^1$  where  $P_0^1$  is obtained based on the activity durations.

**Lemma 2:** For each project with structure of figure 1 and including one phase, if  $DCF^1$  is negative and penalty cost is constant, performing of the project is not economic.

**Proof:** When  $DCF^1 < 0$ ,  $NPV$  increases by delaying the project start time until  $s^1 \leq D - P_0^1$  because by this delay we have postponed the project costs without paying any penalty. Now, if  $p$  is a constant coefficient, the penalty term is defined as  $-p(s^1 + P_0^1 - D)e^{-\alpha(s^1 + P_0^1)}$  for  $s^1 > D - P_0^1$ . In this case, the penalty term is a convex function that its right

part is an increasing function and an asymptotic for zero because  $\lim_{s^1 \rightarrow +\infty} (-p(s^1 + P_0^1 - D)e^{-\alpha(s^1 + P_0^1)}) = 0$ .

Therefore, when  $s^1 \rightarrow +\infty$ ,  $NPV = (DCF^1 \cdot e^{-\alpha s^1} + \text{penalty term}) \rightarrow 0$  that equals to the upper bound of the net present value for a project with negative discounted cash flow. **Lemma 3:** For each project with structure of figure 1 and including one phase, if  $DCF^1$  is negative and penalty cost is computed as the exponential function  $p = \beta e^{\alpha f^1}$  where  $\beta > 0$ , the optimum tardiness of the project is  $s^1 = \alpha^{-1} \cdot \ln\left(-\frac{\alpha \cdot DCF^1}{\beta}\right)$ .

**Proof:** If  $DCF^1 < 0$  and  $p = \beta e^{\alpha f^1}$  where  $\beta > 0$ , the penalty term equals  $-\beta(s^1 + P_0^1 - D)$  which is a decreasing function of  $s^1$ . On the other hand, the  $NPV$  consists of  $DCF^1 \cdot e^{-\alpha s^1}$  which is an increasing function of  $s^1$ . The optimal value of project tardiness equals  $s^1 = \alpha^{-1} \cdot \ln\left(-\frac{\alpha \cdot DCF^1}{\beta}\right)$  that is obtained using

$\frac{\partial NPV}{\partial s^1} = 0$ . If  $\alpha^{-1} \cdot \ln\left(-\frac{\alpha \cdot DCF^1}{\beta}\right)$  is not integer, we round it to the side that gives better  $NPV$ . As we expect to surely finish the project, after this, we consider the penalty term as the exponential function  $p = \beta e^{\alpha f^1}$ .

**Lemma 4:** For each project with structure of figure 1 and including one phase, if  $DCF^1$  is zero, the project should be finished before deadline.

**Proof:** If  $DCF^1 = 0$ ,  $NPV = 0$  for  $\forall f^1 \leq D$ . If  $f^1 > D$ , the penalty term is included in the project costs. Therefore the project should be finished before deadline. Based on these four lemmas, we present the optimal scheduling procedure in table 2.

**Tab. 2. Optimal Scheduling Procedure**

Step1. Compute $DCF^\varphi$ for each phase $\varphi = 1, \dots, Y$ based on the definition 1.
Step2. If $\forall \varphi: DCF^\varphi \geq 0$ , let $f^\varphi = ef^\varphi; \varphi = 1, \dots, Y$ and $f_i = lf_i; \forall i \in A$ .
Step3. If $\exists \varphi_1: DCF^{\varphi_1} < 0$ and $\exists \varphi_2 > \varphi_1: CDCF_{\varphi_1}^{\varphi_2} \geq 0$ , let $f^\varphi = ef^\varphi; \varphi = 1, \dots, \varphi_2$ and $f_i = lf_i; \forall i \in \bigcup_{\varphi=\varphi_1}^{\varphi_2} A^\varphi$ .
Step4. If $\varphi_1$ is the first phase for which $DCF^{\varphi_1} < 0$ and $\neg \exists \varphi_2 > \varphi_1: CDCF_{\varphi_1}^{\varphi_2} \geq 0$ , let $s^{\varphi_1} = \alpha^{-1} \cdot \ln\left(-\frac{\alpha \cdot CDCF_{\varphi_1}^Y}{\beta}\right)$ , $f^\varphi = ef^\varphi; \varphi = 1, \dots, Y$ and $f_i = lf_i; \forall i \in A$ .

Steps 1 and 2 are developed based on the *definition 1* and *lemma 1*, respectively. Steps 3 and 4 consider the situation in which we face a phase  $\varphi_1$  for which  $DCF^{\varphi_1} < 0$ . Step 3 explains for the situation in which  $\exists \varphi_2 > \varphi_1 : CDCF^{\varphi_2}_{\varphi_1} \geq 0$ .

In this case, the NPV decreases if we delay the start of phase  $\varphi_1$  because the partial network including activities and milestones of phases  $\varphi_1$  to  $\varphi_2$  has a non-negative discounted cash flow that based on *lemma 1* should be performed as soon as possible. In step 4 when we face for the first time with the case in which  $\neg \exists \varphi_2 > \varphi_1 : CDCF^{\varphi_2}_{\varphi_1} \geq 0$ , we have to postpone the start of phase  $\varphi_1$  as many as  $\alpha^{-1} \cdot \ln \left( -\frac{\alpha \cdot CDCF^{\varphi_1}}{\beta} \right)$  based on the *lemma 3*.

## 5. Conclusions

In this paper we developed an optimal solution procedure for the project scheduling problem with permitted tardiness and discrete time/resource trade-offs under maximum net present value objective. This problem is very important and commonly used in practice particularly in the construction industries. We proved that the mode corresponding with the minimum possible duration of each activity is the optimal mode for the activity in order to minimize project costs. Then we presented an optimal scheduling procedure for the problem in which activities have fixed modes. Our future research will focus on the same problem with resource constraints or with considering employing costs to model better the real world conditions.

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