Boundary Condition Identification of a Plate on Elastic Support

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The behaviour of mechanical structures in low frequencies is strongly affected by the existence of the boundary conditions. It is not usually possible to provide ideal boundary conditions, i.e. simply supported or clamped, for structures. Therefore the real structures are mostly constrained by elastic supports. Constructing an accurate mathematical or numerical model for a structure requires the knowledge of the support parameters. In this paper, a new method is proposed for the parameter identification of a rectangular plate constrained by elastic support. The method relies on the free vibration solution of the plate dynamics subjected to elastic boundary conditions and employs the optimization toolbox of MATLAB.

1. INTRODUCTION

The supports, or boundary conditions, play an important role in a structure's dynamic behaviour and must be considered carefully when constructing mathematical or numerical models. In reality, the supports of structures are not rigid enough, and they show flexibility to some degree. The flexibility of the supports can be modelled as elastic boundary conditions. In order to have an accurate model of a structure, the knowledge of the support parameters is essential. The support parameters can be identified by using experimental results.

The sensitivity method is one of the most widely used approaches in determining boundary condition parameters.[1] In this method the difference between model predictions and test observations is defined as an objective function. An iterative process is then adopted, and the objective function is minimized by using the sensitivity approach. It should be noted that the sensitivity of higher natural frequencies to support parameters is low, which results in convergence problems in the optimization procedure.[2]

In the characteristic equation method the boundary support parameters are identified by solving the nonlinear characteristic equations. In this method, which was adopted by Ahmadian et al., the number of characteristic equations formed is equal to the number of measured natural frequencies. The boundary condition parameters are then identified by simultaneously solving the characteristic equations.[3]

Waters et al. and Wang and Yang adopted the static flexibility measurements and identified the boundary conditions of a tapered beam.[4,5] They modelled the beam as a uniform rigid beam that was constrained by collocated equivalent translational and rotational springs. The boundary conditions are identified by quasi-static stiffness measurements obtained from impact tests.

This paper deals with the support parameter identification of a rectangular plate constrained in its edges by an elastic boundary condition. The boundary condition contains structural damping. The solution method proposed by Li et al. is adopted to analyse the free vibration of the beam.[6] The analysis leads to obtaining the natural frequencies and damping ratios of the plate. An identification approach is proposed based on the solution presented by Li et al. and by using the measured modal properties (i.e. natural frequencies and damping ratios).[6] The proposed method is verified by using simulated and experimental results. The next section considers the free vibration analysis of an elastically supported plate.

2. PLATE DYNAMICS ON ELASTIC SUPPORT[6]

Figure 1 shows an elastically supported rectangular plate, which is constrained by lateral and torsional springs. It is considered that the elastic boundary condition contains structural damping.

The governing differential equation for the free vibration of the rectangular plate is expressed in Eq. (1):

$$D\nabla^4 w(x,y) - \rho h\omega^2 w(x,y) = 0; \qquad (1)$$

where $\nabla^4 = \partial^4/\partial x^4 + 2\partial^4/\partial x^2\partial y^2 + \partial^4/\partial y^4$, and $w(x,y)$ is the lateral displacement function, ω is the angular frequency and ρ, h and D are mass density, thickness, and bending rigidity of the plate, respectively. The above governing equation is

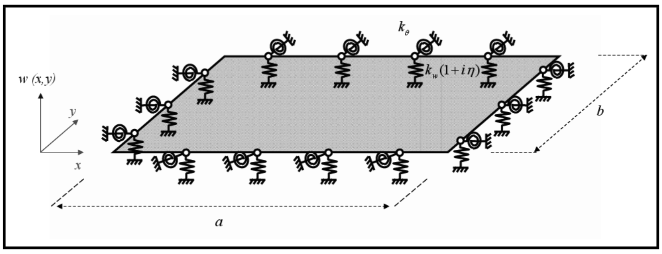

Figure 1. Rectangular plate with elastic boundary condition in all edges

subjected to the following boundary conditions, at $x = 0$ and $x = a$:

$$\overline{k}_w w(x,y) = -D\left(w_{xxx} + (2-v)w_{xyy}\right); \tag{2}$$

$$k_\theta w_x = (-1)^{x/a} D\left(w_{xx} + v w_{yy}\right); \tag{3}$$

at $y = 0$ and $y = b$:

$$\overline{k}_w w(x,y) = -D\left(w_{yyy} + (2-v)w_{yxx}\right); \tag{4}$$

$$k_\theta w_y = (-1)^{y/b} D\left(w_{yy} + v w_{xx}\right); \tag{5}$$

where $k_w = k_w(1+i\eta)$. k_w and k_θ are respectively the lateral and torsional stiffness coefficients of the boundary condition and η is the structural damping coefficient. The above boundary conditions represent the shear forces and bending moments introduced at the plate edges by its movements. By considering a displacement field for the free vibration of the plate, substituting it into the governing equation and boundary condition relations, i.e. Eqs. (1)-(5), the resulting equation can be expressed as:[6]

$$\left([K] - \frac{\rho h \omega^2}{D}[M]\right)\{a\} = 0. \tag{6}$$

Equation (6) can be used for obtaining the natural frequencies of an elastically supported rectangular plate, provided that the parameters of the elastic boundary condition are known. The natural frequencies are calculated by solving the characteristic equation, i.e. $|\mathbf{K} - (\rho h \omega^2/D)\mathbf{M}| = 0$. Since k_w is complex the calculated natural frequencies are a complex conjugate and of the form $\omega_n = -\zeta_n \omega_n \pm i\omega_n$, $n = 1, 2, \ldots$. The real part represents the damping ratio, and the imaginary part is a measure of the free oscillation frequency of each mode.

Equation (6) can also be effectively used for parameter identification of the plate boundary support when the natural frequencies and damping ratios are known from experimental or simulated results. The parameter identification method is discussed in the next section.

3. PARAMETER IDENTIFICATION METHOD

Consider that N natural frequencies and damping ratios of an elastically restrained plate are known,

$$\{\Omega_e\} = [\overline{\omega}_1^e \overline{\omega}_2^e \cdots \overline{\omega}_N^e]^T; \tag{7}$$

$$\{Z_e\} = [\zeta_1^e \zeta_2^e \cdots \zeta_N^e]^T; \tag{8}$$

where $\{\Omega_e\}$ and $\{Z_e\}$ are the vectors of measured or simulated natural frequencies and damping ratios, respectively. The aim of this section is to identify the support parameters, i.e. k_w, k_θ and η by using the known vectors of modal properties introduced in Eq. (7) and Eq.(8). The support parameters are estimated by minimizing the differences between known and predicted modal characteristics as is described in the following. The predicted modal parameters are calculated by using the numerical method presented in the previous section.

In order to start the optimization algorithm, first a set of initial values for the support parameters are considered, i.e. k_w^0, k_θ^0 and η^0. The initial values are updated in subsequent iterations until the optimum support parameters are obtained. By substituting the initial parameters into Eq. (6) and solving the characteristic equation a set of predicted natural frequencies and damping ratios are obtained,

$$\{\Omega_a\} = [\overline{\omega}_1^a \overline{\omega}_2^a \cdots \overline{\omega}_N^a]^T; \tag{9}$$

$$\{Z_a\} = [\zeta_1^a \zeta_2^a \cdots \zeta_N^a]^T; \tag{10}$$

where $\{\Omega_a\}$ and $\{Z_a\}$ are the vectors of the predicted natural frequencies and damping ratios, respectively. The optimum set of the support parameters can be obtained by minimizing the following objective function:

$$OBJ = ||1 - \frac{\{\Omega_a\}}{\{\Omega_e\}}|| + ||1 - \frac{\{Z_a\}}{\{Z_e\}}|| \tag{11}$$

In Eq. (11) OBJ represents the sum of the norm of the differences between the known and predicted natural frequencies and damping ratios. Different optimization algorithms can be used to minimize the objective function of Eq. (11) and hence estimate the optimum support parameters. In sensitivity based approaches, the optimization problem in each iteration is cast in the following first order sensitivity equation: $[S]\{\Delta\} = \{\varepsilon\}$. Here, $[S]$ is the sensitivity matrix, $\{\Delta\} = [\delta k_w, \delta k_\theta, \delta\eta]^T$ is the vector of updating parameters and $\{\varepsilon\}$ is the vector of differences between the known and predicted modal parameters. By solving for $\{\Delta\}$, the updated support parameters in iteration i^{th} are obtained as,

$$k_w^i = k_w^{i-1} + \delta k_w; \tag{12}$$

$$k_\theta^i = k_\theta^{i-1} + \delta k_\theta; \tag{13}$$

$$\eta^i = \eta^{i-1} + \delta\eta. \tag{14}$$

Parameter updating based on equations (12)-(14) is terminated when $||\varepsilon||$ reaches a small value, i.e. $||\varepsilon|| << 1$. $[S]$ is a matrix compose of the sensitivity of different modal parameters to the support parameters. Since the sensitivity matrix is not known for the problem considered in this paper, identification is performed by using gradient based methods in the optimization toolbox of MATLAB, e.g *fmincon*, *fminsearch*, ... In the following section a numerical example is presented to show the accuracy of the proposed method.

Table 1. Mechanical properties of the square plate

v	E(Gpa)	$\rho\ (\frac{\text{kg}}{\text{m}^3})$
0.33	200	7800

Table 2. Simulated natural frequencies and damping ratios.

Mode number	ω(Hz)	ζ (%)
1	115.41	2.04
2	135.43	1.97
3	166.77	1.92
4	217.41	1.71
5	219.97	1.60

4. NUMERICAL EXAMPLE

A square plate of dimensions $a = b$ =2 m and $h = 0.0025$ m is considered which is supported by an elastic boundary condition. The parameters of the boundary condition are considered as $k_w = 1000\ \frac{\text{N}}{\text{m}}$, $k_\theta = 100\ \frac{\text{Nm}}{\text{rad}}$ and $\eta = 0.0005$. The material properties of the plate are given in Table 1,

Having the plate dimensions, its material properties, and the boundary support parameters, the natural frequencies and damping ratios can be calculated by using Eq. (6). Table 2 shows five natural frequencies of the plate and their corresponding damping ratios:

Next, the modal properties presented in Table 2 are considered as experimental results, and the parameters of the boundary condition are identified by minimizing the objective function defined in Eq. (11). Minimization is done by using the Optimization Toolbox of MATLAB. Since the objective function is a nonlinear and complex function, the employed optimization algorithm strongly affects the identified results. The efficiency of different optimization algorithms was studied, and finally it was concluded that the *fmincon* function is the most appropriate function for the minimization of the objective function defined in this paper. In Table 3 the elapsed time and the final value of the objective function for different unconstraint (i.e. *fminsearch* and *fminunc*) and constraint (i.e. *fmincon* and *patternsearch*) optimization algorithms are compared. In obtaining the results presented in Table 3 the initial values shown in Table 4 for support parameters were used. Also, in constraint optimization algorithms, it was considered that the support parameters are positive, i.e. $k_w > 0$, $k_\theta > 0$ and $\eta > 0$.

Results presented in Table 3 indicate that the *fmincon* algorithm is more effective in obtaining the plate support parameters. In Fig. 2 the change in the objective function and, in Table 4, the initial and final support parameters are presented for the *fmincon* algorithm.

Figure 2 shows that the identification algorithm succeeds in finding the optimum support parameters after 40 iterations. The results presented in Fig. 2 and Table 4 indicates that the proposed method identifies the support parameters with an acceptable accuracy. The next section considers two experimental case studies.

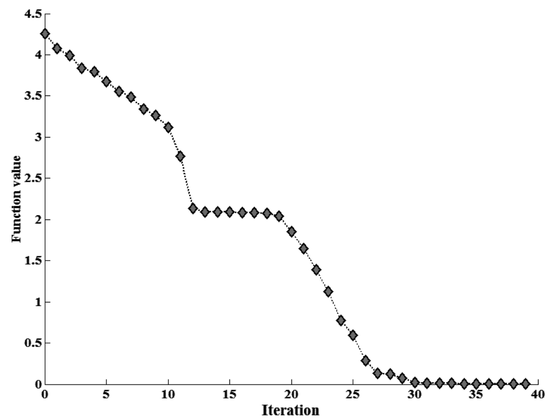

Figure 2. Change in objective function.

Table 4. Initial and identified support parameters.

	$k_w\ (\frac{\text{N}}{\text{m}})$	$k_\theta\ (\frac{\text{Nm}}{\text{rad}})$	η
Initial	0.1	0.0001	0
Numerical example			
Identified	1000.18945	97.0673	0.00049925
Aluminum plate			
Identified	65.104	0.126	0.00028
Steel plate			
Identified	3194.161	850.725	0.01849

5. EXPERIMENTAL VALIDATION

In this section, the proposed method is applied to two experimental case studies, and the parameters of their boundary conditions are identified.

5.1. Aluminium rectangular plate

In this section the experimental results of a rectangular aluminium plate considered by Amabili is used, and its boundary support parameters are identified.[7] The material properties and geometrical dimensions of the aluminium plate are presented in Table 5.

The plate was placed between rectangular frames made of thick steel. The frame prevents the edges of the plate to move in a perpendicular direction, but they can rotate. Therefore the boundary condition was very similar to the simply supported boundary condition. It should be noted that the identified lateral stiffness coefficient should be much larger than the identified torsional stiffness coefficient. Modal testing was performed on the plate, and its natural frequencies and damping ratios were extracted. The plate was excited by means of an electromagnetic shaker, model LDS V406. The transmit-

Table 5. Dimensions and mechanical properties of the aluminum plate.[7]

v	E (Gpa)	$\rho\ (\frac{\text{kg}}{\text{m}^3})$	h (m)	b (m)	a (m)
0.33	69.10	2700	0.0003	0.184	0.515

Table 3. The efficiency of different optimization algorithms.

	fminsearch	*fminunc*	*fmincon*	*patternsearch*
elapsed time (s)	12423.50	736.33	450.73	18663.19
final objective function	2.08×10^{-9}	27.304	2.93×10^{-18}	1.53×10^{-8}

Table 6. Comparison of experimental and predicted modal properties (aluminum plate).

Mode	Natural frequency ω (Hz)			Damping ratio ζ (%)		
number	Exp.	Predicted	Error (%)	Exp.	Predicted	Error (%)
1	26.87	26.99	-0.47	2.06	2.06	0.0
2	39.37	38.97	1.01	1.51	1.50	0.54
3	55.20	55.60	-0.74	1.88	1.90	-1.13
4	75.72	74.32	1.84	1.34	1.31	2.04
5	93.56	95.10	-1.65	1.12	1.13	-1.72

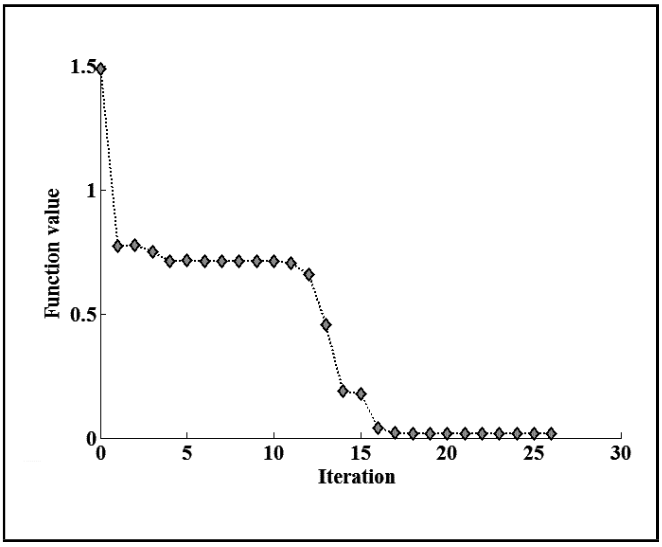

Figure 3. Variation of the objective function (aluminium plate).

ted value was measured by using a piezoelectric force transducer, model PCB M209C11, placed between the stinger and the plate. An accelerometer, model Endevco 22, was glued to the centre of the plate in order to measure the plate response. A low level burst-random excitation force was employed, and the plate frequency response functions (FRFs) were measured. The plate modal characteristics were extracted by analysing the experimental FRFs. The experimental results are presented in Table 6:

The measured modal properties presented in Table (6) are used, and the parameters of the plate boundary support are identified. Identification is done by following the procedure presented in previous sections. It is worth mentioning that only three first natural frequencies are used in the identification procedure. The remaining two natural frequencies are used for the verification of the identified parameters. The variation of the objective function in the identification procedure is shown in Fig. 3. The initial and identified support parameters are tabulated in Table 4. In Table 6 experimental and identified modal properties are compared. The results presented in Table 6 show the accuracy of the proposed method.

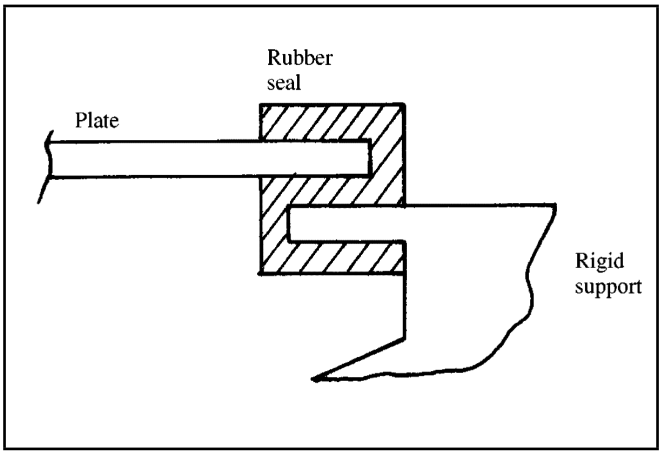

Figure 4. The steel plate supported by rubber seal.

5.2. Steel plate supported by rubber seal

In this section, the boundary condition of the steel plate considered by Ahmadian et. al. is identified.[3] The plate, having the dimensions of 0.5 m $\times$ 0.8 m $\times$ 0.0025 m, is attached to the ground by the rubber seal. A schematic of the elastic supported plate is depicted in Fig. 4. The plate has the following material properties: $v = 0.33$ as Poisson's ratio, $E = 207$ GPa as Young's modulus, and $\rho = 7800 \frac{\text{kg}}{\text{m}^3}$ as mass density.

Modal testing was performed on the plate in order to measure its dynamic properties, i.e. natural frequencies and damping ratios. The plate was excited by using a modal hammer, and its response was measured by means of accelerometers. By transfering the recorded force and response signals into a digital analyser, the frequency response functions (FRFs) were calculated. FRFs were then curve fitted, and the modal parameters of the steel plate were extracted. The experimental natural frequencies and damping ratios are given in Table 7. As in the aluminium plate case, the elastic support parameters are identified by employing the method presented in this paper and by using the first three measured natural frequencies and damping ratios. The variation of the objective function in the identification procedure is shown in Fig. 5. Table 4 reports the identified support parameters for the steel plate.

The experimental and predicted modal properties are compared in Table 7. It is worth mentioning that the last two sets of

Table 7. Comparison of the experimental and predicted modal properties.

Mode	Natural frequency ω (Hz)			Damping ratio ζ (%)		
number	Exp.	Predicted	Error (%)	Exp.	Predicted	Error (%)
1	29.60	29.89	-1.00	1.58	1.57	0.32
2	60.40	59.93	0.77	0.84	0.85	-1.23
3	98.90	97.80	1.11	0.91	0.92	-1.56
4	106.20	104.50	1.53	0.87	0.85	1.43
5	120.10	122.90	-2.41	1.36	1.39	-2.72

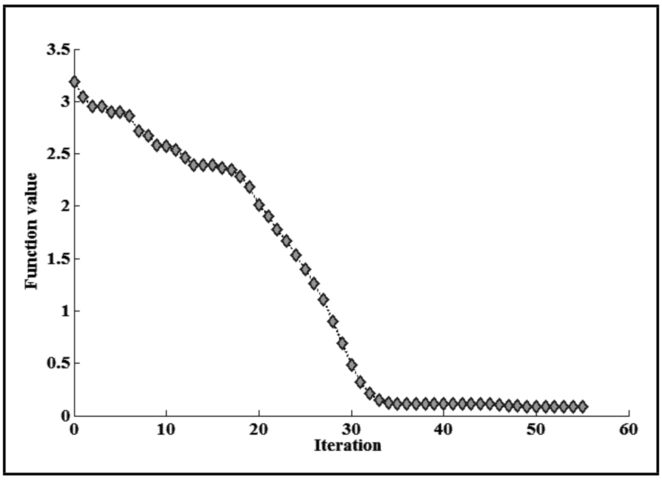

Figure 5. Variation of the objective function in identification procedure (steel plate).

modal characteristics are used for verification of the identified model. The results presented in Table 7 show the accuracy of the identified support model for the steel plate.

6. CONCLUSION

Identification of the boundary condition parameters of a rectangular plate restrained in edges by an elastic support was considered. The boundary support was considered to contain structural damping. In order to identify the support parameters, first a numerical solution developed in[6] was presented for free vibration of elastic supported plate. The solution permitted the calculation of the plate's natural frequencies and damping ratios. The support parameters were identified by minimizing the differences between experimental and predicted modal properties by employing the MATLAB optimization toolbox. The identification procedure was verified by using simulated and experimental results presented by Amabili[7] and Ahmadian et. al.[3]

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