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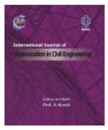

UPPER AND LOWER BOUNDS FOR THE COLLAPSE LOAD FACTOR OF RECTANGULAR GRIDS USING FEM

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ABSTRACT

Despite widespread application of grillage structures in many engineering fields such as civil, architecture, mechanics, their analysis and design make them more complex than other type of skeletal structures. This intricacy becomes more laborious when the corresponding analysis and design are based on plastic concepts.
In this paper, Finite Element Method is utilized to find the lower and the upper bounds solutions of rectangular planner grids and this method is compared with analogues Finite Difference Method to indicate the efficiency of proposed approach.

Keywords: grid structures, plastic design, finite element method, finite difference method

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1. INTRODUCTION

Nowadays, the efficiency of analysis and design plays a key role in engineering projects and the method selection indubitably is one of the important steps of increasing of efficiency.

There are two general categorized methods to analysis and design of structures. If the compatibility equations (Displacement Conditions), the Equilibrium equations (Static Conditions) and the force-displacement relationships are employed for the analysis, then it is called an elastic analysis. However, if the utilized tools are the mechanism conditions, the equilibrium conditions and the yield condition, then the method is known as the plastic analysis and design.

The initial steps to develop plastic analysis and design concepts were taken by Baker and Heyman [1]. In the following Horne [2] and Chen [3] developed the fundamental theorems and concepts by publishing their books, articles and thesis. As recent published books in this scope one can mention Wong [4] efforts that include some novel subjects like incremental

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analysis of frames.

As stated, efficiency of any selected method for analysis and design is the most important feature of the synthesis procedures. In the recent researches, especially in tackling with real-size structures, the combination of optimization methods and classic approaches lead to interesting outcomes. Kaveh and Mokhtar-zadeh [5] developed optimal plastic analysis and design of frames based on graph theoretical methods for the first time. After that the genetic algorithm was employed in order to find collapse load factor for rigid-plastic analysis of frames by Kaveh and Khanlari [6]. Kaveh and Jahanshahi extended the application of meta-heuristics and ant colony algorithms to plastic analysis and design of frames [7-8].

Ant colony and charged system search algorithms were employed to achieve the optimum elementary mechanism of frame structures which were generated by combination method for planner frames [9]. A very complete set of application of metaheuristic algorithms in optimization is released as [10] and optimal analysis and design of structures is discussed in [11]. In other hand, the graph based optimization methods collected in [12,13].

Among many applications of plastic methods, the analysis and design of grillages is highly appealing. Simplification of complex procedure of grids analysis, convenience and accuracy in calculations are some of the features lead to choice of plastic synthesis as a suitable tool for the analysis and design of this type of structures. Grillages are type of structures in which straight members resist in the same plane and loads applied in perpendicular to it. Where the members cross each other, they are taken to be joined with full strength connections so that any required forces and moments can be transmitted through the structure. Grid structures have widespread usage in many fields of civil engineering as application in composites structures, bridges, etc., in architecture as application in domes, spatial design, etc. and many other fields as the scope of application of grids.

In this paper after description of fundamental theorems of plasticity, grids equilibrium equations, FDM and FEM formulations, a Finite Element Method procedure (FEM) is presented in order to calculate the lower and the upper bounds of rectangular planner grids and this presented procedure is compared with a Finite Difference Method (FDM) procedure of Gregorian [14], in CPU Time Criteria based.

2. FUNDAMENTAL THEOREMS OF PLASTICITY

In order to find an explicit relation for plastic analysis and design of grid structures, at the incipient stage it is necessary to review fundamental theorems of plasticity.

2.1. Lower Bound Theorem (Safe Theorem)

If, at any load factor λ, it is possible to find a bending moment distribution in equilibrium with the applied loads and everywhere satisfying the yield condition, then λ is either equal to or less than the load factor at failure [2].

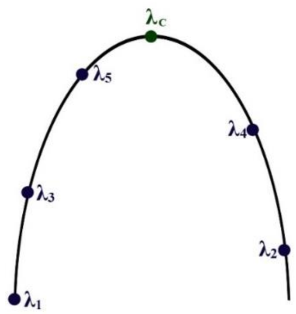

Fig. 1 The plastic collapse load factor and the load factors are found by using lower bound theorem.

2.2. Upper Bound Theorem (Unsafe Theorem)

If for any assumed plastic mechanism, the external work done by the loads at a positive load factor λ is equal to the internal work at the plastic hinges, then λ is either equal to or greater than the load factor at failure [2].

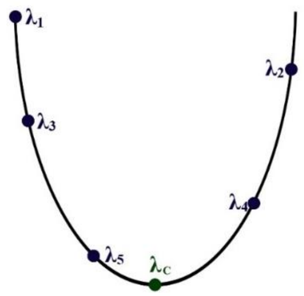

Fig. 2 The plastic collapse load factor and the load factors are found by using upper bound theorem.

2.3. Uniqueness Theorem

If, at any load factor λ, a bending moment distribution can be found which satisfies the three conditions of equilibrium, mechanism, and yield, then that load factor is the collapse load factor λ_c [2].

3. EQUILIBRIUM EQUATIONS OF GRID

An engineering structure must be strong enough to resist its applied loads without collapsing, and stiff enough not to deflect unduly under external loading [1]. In the classic standpoint (it can be interpreting as linear elastic analysis) it is necessary to employ the displacement condition as Compatibility Equations, the Statics condition as Equilibrium Equations and the Moment-Curvature Relation ($M = +EIK$). In other hand, in plastic analysis and design, one can ignore elastic deflections and use different tools known as Mechanism, Equilibrium and Yield Conditions ($|M| \le M_p$). In order to analysis a grid structure it is necessary to use these three basic tools in addition to fundamental plasticity theorems.

As mentioned, grids consist of straight members is a plane and the external loads are imposed perpendicular to this plane. It can be proven that the effect of twisting moments can be neglect in comparison to bending moment in uniform symmetric rectangular grids [14].

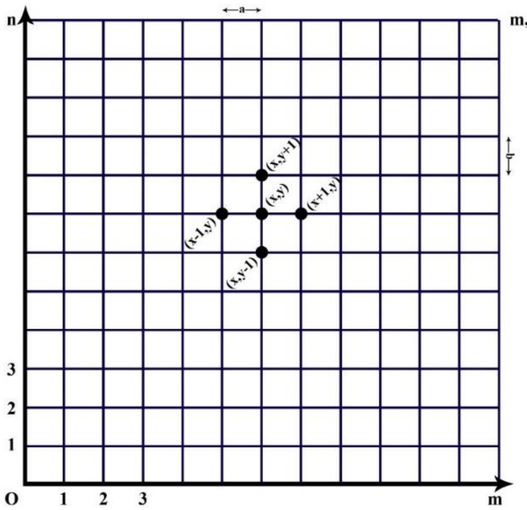

Fig. 3 Schematic of a uniform rectangular grid.

The lower bound solution of a uniform grid in which simple supports form the boundary conditions and the structure subjected to uniformly distributed normal nodal forces, can be obtained from the equilibrium equations.

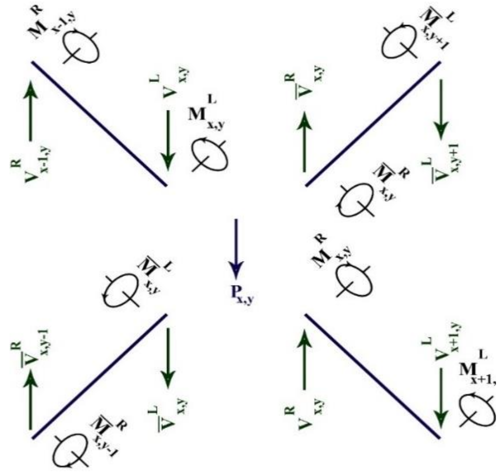

Fig. 4 Nodal Free-body Diagram in a Uniform Grid.

By using the three mentioned tools it will be possible to develop the characteristic equations of this type of grids. In any point of a grid, it is possible to consider a bending moment distribution in the X and Y directions in which at any point the moment not violate the yield criterion as bellow:

$$M_{xy} = \alpha M(mx - x^2) \tag{1}$$

$$\overline{M}_{xy} = \overline{\alpha}\overline{M}(ny - y^2) \tag{2}$$

where the M_{xy} and $\overline{M}_{xy}$ denote fully plastic bending moments of the grid in the X and Y directions, respectively. Here, m and n also, denote number of cells in the X and the Y direction, respectively. In relations 1 and 2, α and $\overline{\alpha}$ are defined as follows:

$$\alpha = \frac{4}{m^2 + \delta_m} \tag{3}$$

$$\bar{\alpha} = \frac{4}{n^2 + \delta_n} \tag{4}$$

In which

$$\delta_m = \frac{(-1)^m - 1}{2} \tag{5}$$

$$\delta_n = \frac{(-1)^n - 1}{2} \tag{6}$$

The stated conditions of equilibrium may be written as follows:

$$\sum M_{xy} = 0 \Rightarrow M_{xy}^{Right} - M_{xy}^{Left} = 0 \tag{7}$$

$$\sum \overline{M}_{xy} = 0 \Rightarrow \overline{M}_{xy}^{Right} - \overline{M}_{xy}^{Left} = 0 \tag{8}$$

$$\sum F_y = 0 \Rightarrow V_{xy}^R - V_{xy}^L + \overline{V}_{xy}^R - \overline{V}_{xy}^L + P_{xy} = 0 \tag{9}$$

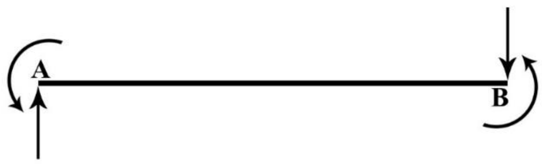

Fig. 5 Static equation in far Node (B) from applied Node (A).

According to Figure 5, it is possible to employ static equation in node B.

$$\sum M_B = 0 \Rightarrow M_A + M_B - V_A L = 0 \tag{10}$$

Application of Eq. (10) into nodal equilibrium (like Figure 4) leads to Eqs. (11-14) as follows:

$$V_{xy}^R = \frac{1}{a}(M_{(x+1),(y)}^L - M_{xy}^R) \tag{11}$$

$$V_{xy}^L = \frac{1}{a}(M_{xy}^L - M_{(x-1),(y)}^R) \tag{12}$$

$$\overline{V}_{xy}^R = \frac{1}{b}(\overline{M}_{(x+1),(y)}^L - \overline{M}_{xy}^R) \tag{13}$$

$$\overline{V}_{xy}^L = \frac{1}{b}(\overline{M}_{xy}^L - \overline{M}_{(x-1),(y)}^R) \tag{14}$$

4. FINITE DIFFERENCE FORMULATION OF GRIDS

The finite difference method (FDM) is a powerful approach in order to solve differential equations. In this numerical method it is possible to approximate the derivatives by using finite differences when the function is only known at discrete points. In other word, in a continuous function there is infinitive amount of information which is not possible to store all of them in a calculation. However, if it is possible to store the values of the function at discrete points then the missing information can be detected. The most important issue in choosing the number of points is to balance the accuracy of derivatives while not becoming computationally excessive.

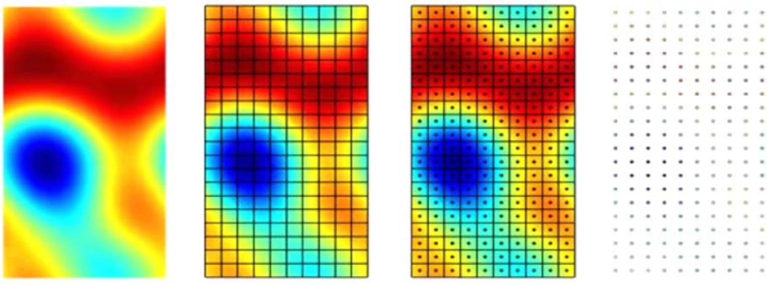

Fig. 6 Finite Difference method steps.

There are three types of finite difference approximations. In central difference approximation derivatives are calculated from mid-point between two adjacent points. This is the most accurate one because the information from both sides is being used. Equation (15) indicates this type of formulation.

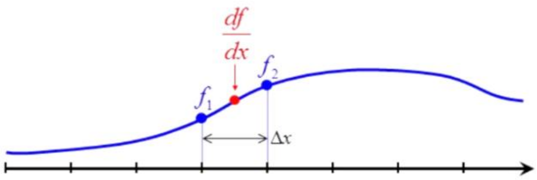

Fig. 7 Central Difference Approximation in FDM.

$$\frac{df}{dx} \approx \frac{f_2 - f_1}{\Delta x} \tag{15}$$

That is possible to use backward and forward difference approximations when the information that is needed in the previous type are not accessible. Equations (16) and (17) (corresponding to Figs. 8 and 9) demonstrate these types of FDM approximations.

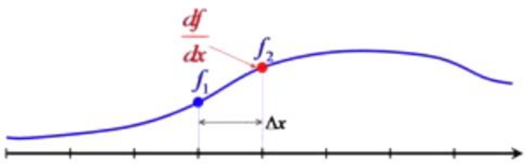

Fig. 8 Backward Approximation in FDM.

$$\frac{df}{dx} \approx \frac{f_2 - f_1}{\Delta x} \tag{16}$$

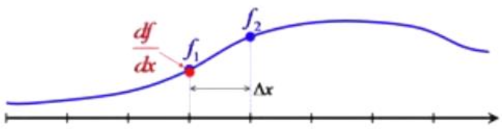

Fig. 9 Forward Approximation in FDM.

$$\frac{df}{dx} \approx \frac{f_2 - f_1}{\Delta x} \tag{17}$$

In order to generalize this, it is possible to approximate any order derivative at any position as a linear weighted sum of known function values.

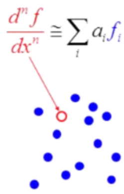

Fig. 10 High order approximation in FDM.

The difficult part is to determine the weights to have accurate derivatives. At first, it is needed to choose the order of accuracy (N) by determining the number of points for approximating the derivatives. Thus, it can be written as a polynomial of order N by N+1 unknowns which are called the polynomial coefficients.

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + ... + a_N x^N \tag{18}$$

This polynomial must be written at each of discrete points, leading to Eq. (19).

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^N \\ 1 & x_2 & x_2^2 & \dots & x_2^N \\ 1 & x_3 & x_3^2 & \dots & x_3^N \\ 1 & \vdots & \vdots & \dots & \vdots \\ 1 & x_{N-1} & x_{N-1}^2 & \dots & x_{N-1}^N \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \square \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-1}) \end{bmatrix} \tag{19}$$

By solving Eq. (19), one can determine the coefficients that satisfy the polynomial equation.

To achieve a lower bound solution for plastic analysis of 2D grid, finite difference technique can be used to formulate the equations of equilibrium. It leads to the governing difference equation of the equilibrium as Eq. (20), [14].

$$(\Delta_x - \nabla_x)\frac{M_{xy}}{a} + (\Delta_y - \nabla_y)\frac{\overline{M_{xy}}}{b} = -P_{xy} \tag{20}$$

where the symbols Δ and ∇ are the first forward and backward difference operators, respectively.

In the limit as the mesh size (a×b) becomes infinitely small, this equation coincides with the corresponding plate equation.

$$\frac{\partial^2}{\partial x^2} M_{xy} + \frac{\partial^2}{\partial y^2}\overline{M_{xy}} = -q_{xy} \tag{21}$$

5. FINITE ELEMENT FORMULATION OF GRIDS

In finite element method, the elements of a grid are assumed to be rigidly connected. So that, the original angles between elements connected together at a node remain unchanged. Both torsional and bending moment continuity then exist at the nodal points of a grid. A typical grid structure subjected to loads F_1 , F_2 , F_3 and F_4 is shown in Fig. 11.

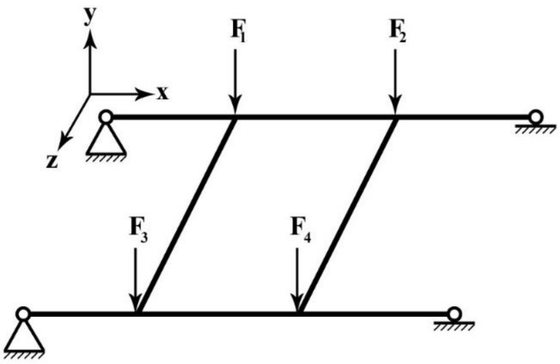

Fig. 11 Typical grid structure applied perpendicular external loads.

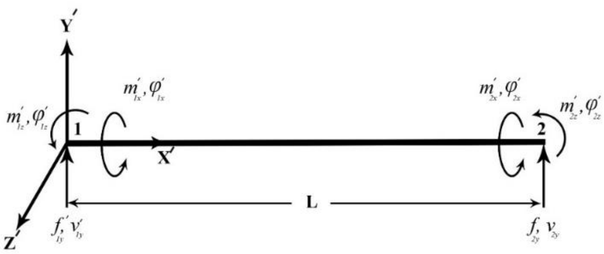

Fig. 12 Grid element with nodal degrees of freedom and nodal forces.

The characteristic Eq. (21) could be obtained using FEM equilibrium equation as Eq. (22).

$$\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{2y} \\ m'_{2x} \\ m'_{2z} \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & \frac{GJ}{L} & 0 & 0 & \frac{-GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{4EI}{L} & \frac{-6EI}{L^2} & 0 & \frac{2EI}{L} \\ \frac{-12EI}{L^3} & 0 & \frac{-6EI}{L^2} & \frac{12EI}{L^3} & 0 & \frac{-6EI}{L^2} \\ 0 & \frac{-GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & \frac{-6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} v'_1 \\ \phi'_{1x} \\ \phi'_{1z} \\ v'_2 \\ \phi'_{2x} \\ \phi'_{2z} \end{Bmatrix} \tag{22}$$

where E, I, L, G and J are elasticity modulus, moment of inertia, length of element, shear modulus and polar moment of inertia, respectively.

6. KINEMATICS AND CHARACTERISTIC EQUATIONS OF GRIDS

The work equation related to current problem can be written as Eq. (23).

$$4\sum_{x=a}^{\frac{a}{2}(m-1)} \sum_{y=b}^{\frac{b}{2}(n-1)} PZ_{xy} = 4\sum_{y=b}^{\frac{b}{2}(n-1)} M\theta_{xy} + 4\sum_{y=a}^{\frac{a}{2}(m-1)} \overline{M}\varphi_{xy} \tag{23}$$

$$Z_{xy} = \frac{4xy}{(m-1)(n-1)ab} w \tag{24}$$

where a and b are the length of cells in directions X and Y, respectively. In Eq. (23), θ_{xy} and ϕ_{xy} are defined as:

$$\theta_{xy} = \frac{\partial}{\partial x} Z_{xy} \tag{25}$$

$$\phi_{xy} = \frac{\partial}{\partial y} Z_{xy} \tag{26}$$

Now,

$$\frac{\partial^2}{\partial x^2} M_{xy} + \frac{\partial^2}{\partial y^2} \overline{M_{xy}} = -q_{xy} \tag{27}$$

7. NUMERICAL EXAMPLES

Figures 13 and 14 illustrate an application of grillage structures in which there is a grid with 12×12 cells.

Fig. 13 An application of the grid structures [15].

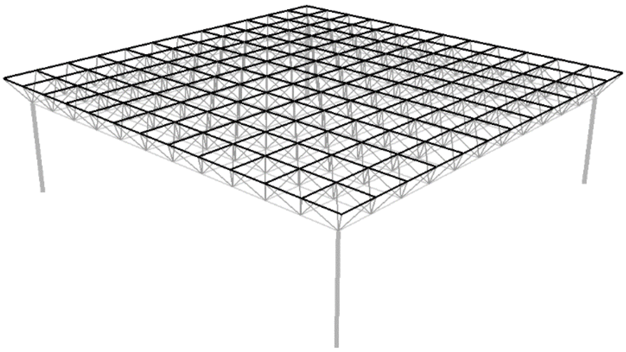

Fig. 14 Schematic mesh model of Figure 13.

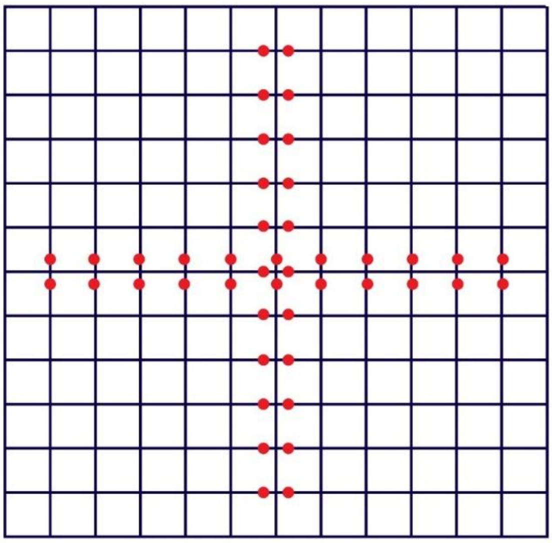

Fig. 15 Collapse mechanism of the grid shown in Figure 14.

The collapse load corresponding to collapse mechanism that illustrated in Fig. 15 can be obtained by using FEM procedure as:

$$p_{Collapse} = \frac{1}{9} M_{Plastic}$$

This calculated collapse load is verified by related researches provided in reference list. Some other instances can be found in Table 1 in which different type of grids are solved by FEM formulation (explained in Section 5).

Table 1: Different examples of the grid structures and their collapse load solution using FEM.

No.	Size of Grid	Collapse Load	Type of Deformation	FEM Time Per FDM Time
1	720 × 720	$P_{Collapse} = \frac{1}{32400} M_{Plastic}$	Pyramid	$\frac{0.02546}{0.03475} = 0.733$
2	3850 × 3850	$P_{Collapse} = \frac{4}{3705625} M_{Plastic}$	Pyramid	$\frac{0.09862}{0.12365} = 0.798$
3	2000 × 1000	$P_{Collapse} = \frac{1}{100000} M_{Plastic}$	Frustum	$\frac{0.07182}{0.09624} = 0.746$
4	6000 × 5000	$P_{Collapse} = \frac{61}{112500000} M_{Plastic}$	Frustum	$\frac{1.52364}{2.22292} = 0.685$

8. CONCLUSIONS

In this paper, a finite element method is presented in order to calculate the lower bound solution based on equilibrium equations and the upper bound solution based on kinematics equations for 2D rectangular grids. By using appropriate distribution of moment the lower and the upper bound solution has become equal leading to the plastic collapse load factor based on uniqueness theorem**.** Finally, to indicate the efficiency of the proposed method, it is compared with analogues finite difference method. As illustrated in Table 1, the advantage of the proposed procedure is about 25 percent in quasi-small structures. By increasing the size of the grids, the efficiency of presented method also enhances and in quasi-large grids the improvement of computational time increases to 32 percent. Therefore, it can be concluded that the outcome of current paper is an efficient procedure.

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