1 Introduction

Small hydropowers are important especially in remote areas. However, the cost per kilowatt of the energy produced by these plants is higher than large hydropower plants. Using reverse pumps is an alternative to overcome this problem. Reverse pumps have been successfully applied in a wide range of small hydrosites in the world. A turbine will normally show a higher efficiency in turbine mode than a pump that is designed for the same operating conditions. On the other hand, the best efficiency of a reverse pump is almost the same as that of the pump mode [1–3]. Therefore, the application of reverse pumps in larger capacities is not economical. By increasing the overall efficiency of reverse pumps, they can be economically applied in the higher power capacities.

Lueneburg and Nelson [4], Williams [5], Cohrs [6], Hirschberger and Kuhlmann [7], and Singh [8] reported efficiency improvement by some modifications on pump components. The aim of this study was the optimization of the impeller geometry of a pump to improve its turbine mode maximum efficiency. An old impeller can be replaced by a new one to reach higher efficiency in turbine mode. A pump manufacturer will produce two impellers for its centrifugal pumps: one for pump running and another for turbine running. The additional modification proposed by Singh [8] can be done on the other component of the pump (i.e., volute and impeller clearance).

In this paper, shapes of the blades were redesigned using a gradient based optimization method involving incomplete sensitivities for radial turbomachinery developed by Derakhshan et al. [9] to obtain higher efficiency. Optimization program was coupled to FINE/TURBOP v.7, to solve 3D incompressible Navier-Stokes equations, and AUTOGIDS mesh generator developed by Numeca. In the next step, the optimized impeller was modified by rounding of leading edges and hub/shroud inlet edges in turbine mode. After each modification, a new impeller was manufactured and tested in the test rig. The efficiency was improved in all measured points by the optimal design of the blade and additional modification as the rounding of the blade’s profile in the impeller inlet and hub/shroud inlet edges in turbine mode. Experimental results confirmed the numerical efficiency improvement in all measured points. This study illustrated that the efficiency of the pump in reverse operation can be improved just by impeller modification. [DOI: 10.1115/1.3059700]

Keywords: blade rounding, blade shape optimization, efficiency improvement, reverse pump, small hydrosite

2 Efficiency Improvement of Centrifugal Reverse Pump

The main objective was to reach higher efficiency by redesigning the blades. Using the gradient based optimization algorithm and incomplete sensitivities method developed by Derakhshan et al. [9], the shape of the blades was redesigned. In the next step, the optimized impeller was modified by rounding of leading edges and hub/shroud inlet edges in turbine mode.

The following steps were done to modify the pump impeller to reach higher efficiency in turbine mode.

2.1 3D Shape Optimization of the Blade. The general form of a shape optimization problem can be written as [10]

\[ \min J(x_c, W(x_c), \nabla_x W(x_c)) \]

\[ S(x_c, W(x_c), \nabla_x W(x_c)) = 0 \]

where \( x_c \) is the control variable for the shape, \( W \) is the flow variable, \( S \) is the state equation, \( g_1 \) is the geometrical constraints, \( g_2 \) is the state constraints, and \( J \) is the cost function that should be minimized.

The local optimization algorithm can be summarized as follows.

For the optimization loop,

1. Provide initial shape parametrization, \( x_0^k \).
2. For \( k = 2, 3, \ldots, k_{\text{max}} \), do:
3. Compute the flow state: \( W(x_c^k) \).
4. Compute the cost function: \( J(x_c^k, W(x_c^k)) \).
5. Compute the incomplete sensitivity of the cost function: \( dJ(x_c^k, W(x_c^k))/dx_c \).
6. If \( \left| (dJ(x_c^k, W(x_c^k))/dx_c) < \varepsilon \right| \) or \( J(x_c^k, W(x_c^k)) < \varepsilon \), STOP.
7. Compute \( x_{c+1}^k \) minimizing \( J \) using incomplete gradient and the approximate inverse of Hessian by BFGS (Broyden, Fletcher, Goldfarb, and Shanno) and evaluating when it is necessary \( W(x_c^{k+1}) \) and \( J(x_c^k, W(x_c^k)) \).
2.1.1 Shape Parametrization. Several parametrizations are possible to describe aerodynamic or hydrodynamic shapes. In radial turbomachinery, one can be to consider the spanwise blade angle distribution from leading to trailing edges. The performance of a radial turbomachine (i.e., centrifugal pump) is intensely influenced by these blade angles [11]. Previous results have shown that hydraulic efficiency is not sensitive to small perturbations in blade thicknesses [11]. On the other hand, thickness is one of the manufacturing constraints. Therefore the blade thicknesses were frozen in the optimization process. Also in this optimization, other manufacturing constraints were ignored. The meridian plane of the hub and shroud and the outlet diameter (in centrifugal pump) were fixed. The optimization was performed in two steps.

2.1.2 Primal Optimization. For a radial blade, the camber lines in hub-span, midspan, and shroud-span were linked through the following relation:

$$d\theta(R) = \theta(R) - \theta_{\text{initial}}(R) = \frac{c_1(R - R_1) + c_2(R - R_1)^2 + c_3(R - R_1)^3}{R} \quad (2)$$

where $\theta$ is the tangential angle, $R$ is the blade radius, and $R_1$ is the radius of the blade leading edge in hub-span (in centrifugal pump). The profile is fixed in $R_1$. The coefficients $c_1$, $c_2$, and $c_3$ are control parameters (Fig. 1(a)).

2.1.3 Final Optimization. The optimal shape from primal parametrization was used as initial guess for the second level parametrization. The camber lines of midspan and shroud-span were linked to the blade camber line in hub-span through

$$d\phi(i) = \frac{c_{1i} + 2S + c_{3i}S^2 + c_{5i}S^3}{S} \quad (3)$$

where $i = 2, 3, m = \frac{i}{R}$, and $dm = \sqrt{R^2 + dx^2}$, in $m$-conformal plan $s = \sqrt{m^2 + \theta^2}$. Here the blade camber lines in midspan and shroud-span are rotated around trailing edge, which is fixed with respect to the hub-span camber line of the blade (Fig. 1(b)). The coefficients $c_{1i}$, $c_{3i}$ and $c_{5i}$ are control parameters for midspan and $c_{1i}$, $c_{3i}$ and $c_{5i}$ for shroud-span.

2.1.4 Sensitivity and Incomplete Sensitivity. The gradient of a cost function $J(x, q(x), W(q(x)))$, function of shape control parameters $x_c$, geometric entities $q(x_c)$ (normal, volume, surface, etc.), and state variables $W$ can be derived using chain rule

$$\frac{dJ}{dx_c} = \frac{\partial J}{\partial x_c} + \frac{\partial J}{\partial q} \frac{\partial q}{\partial x_c} + \frac{\partial J}{\partial W} \frac{\partial W}{\partial q} \frac{\partial q}{\partial x_c} \quad (4)$$

where $f$ and $g$ are functions involving geometrical quantities and state quantities, respectively. For incomplete sensitivity application, the cost function should be expressed as a function of the aerodynamic coefficients or more generally [12]

$$J = \int f(x, q(x))g(W(q(x))) \, d\gamma \quad (5)$$

The dominant part in the gradient comes from geometrical quantities and not from state linearization [12]. More precisely, the last term in gradient expression can be neglected:

$$\frac{dJ}{dx_c} = \frac{\partial J(W)}{\partial x_c} + \frac{\partial J(W)}{\partial x_c} \frac{\partial x_c}{\partial x_c} \quad (6)$$

This gradient approximation avoids the evaluation of an adjoint state and decreases the computational cost. Typical functionals in this class are aerodynamic or hydrodynamic forces on a shape along an arbitrary direction as

$$T_c = \left( \int_{\Gamma} [T \cdot n] \, d\Gamma \right) \cdot \sigma \quad (7)$$

where $T = -pI + (\nu + \nu)(\nabla u_t + \nabla u_t)$ and $n$ is the normal to the shape, $\sigma$ is an arbitrary direction, and $T$ is the Newtonian stress tensor.

In this optimization, incomplete sensitivities were improved by adding supplementary terms to add physical sense to the approximate gradient. In other words, reduced order models (i.e., wall functions) can improve incomplete sensitivity in an inexpensive way. The method and its formulation can be found in the Appendix and in our previous work [9].

2.1.5 Cost Function and Its Reformulation. The aim of this study was blade shape optimization of pump impeller in reverse operation to reach higher efficiency in its rated point defined as

$$\eta_h = \frac{T_{\omega h}}{\gamma q h} \quad (8)$$

where $h$ is head (m), $q$ is flow rate $(m^3/s)$, $T_{\omega}$ (Nm) is the axial torque from the fluid to the impeller, $\omega = \frac{2\pi n}{60}$, and $\gamma$ is specific gravity $(kg/m^3 s^2)$.

To use incomplete sensitivities, the cost function must be based on information of the shape (or part of it). In Eq. (8) increasing the torque improves the efficiency $J = -T_c/T_{\omega h}$, where

$$T_c = \left( \int_{\Gamma} [T \cdot n] \, d\Gamma \right), \quad T_{\omega} = -pI + (\nu + \nu)(\nabla u_t + \nabla u_t)$$

However one may need to improve the hydraulic efficiency of the pump at constant design point (constant specific speed). So
looking for higher hydraulic efficiency should be done at the
given head (total pressure difference between outlet and inlet) and
flow rate. The flow is constant in the optimization process and can
be imposed through boundary conditions. Head (or pressure dif-
fERENCE) can be added as a penalty in the cost function

\[ J = - \frac{T_c}{T_{c0}} + \alpha \frac{[\beta - h_0]}{h_0} \]  

(9)

Unfortunately, this new term does not enter into the incomplete
sensitivity validity domain as it is defined away from the shape
and also does not include any geometric quantity. Eventually,
the cost function accounting for constant head can be reformulated to
adapt the incomplete sensitivity method using reformulated pres-
sure difference (or head) based on axial and radial forces and
blade volume in radial turbomachinery

\[ J = - \frac{T_c}{T_{c0}} + \alpha \frac{[F_{\alpha} - F_{\alpha0}]}{F_{\alpha0}} + \beta \frac{[F_R - F_{R0}]}{F_{R0}} + \gamma \frac{[V_b - V_{b0}]}{V_{b0}} \]  

(10)

which enters the incomplete sensitivity validity domain. Indeed,
the cost function is the rotor axial force with state constraints on
hydrodynamic axial \((F_{\alpha})\), radial forces \((F_r)\), and geometrical
constraint on the blade volume \((V_b)\). The details of this reformulation
can be found in the authors’ previous work [9].

Hydraulic efficiency will be improved by increasing torque and
at the same time keeping \(F_{\alpha}, F_R\), and \(V_b\), and therefore the head
unchanged.

Indeed, one could have used \(x^2\) for penalty, but in this case the
conditioning of the problem degrades. To illustrate this point, con-
sider that \(|x|\) and \(x^2 - |x|\) do not change the condition number, but is
not differentiable. Therefore, it would be best to use a regularized
\(|x|\), which is \(x\) away from 0 and \(x^2\) close to 0. We used the follow-

\[ \text{absr}(x) = \frac{x^2}{|x| + e_1} \]  

\[ = \frac{|x| + e_2}{|x| + e_1} \]

(11)

Figure 2 shows the behavior of the different functions near 0. The
\text{absr}(x) remains differentiable near 0 and has the same behavior
with \(x^2\) away from the origin. In the sequel, we used regularized
\(|x|\) in the functional.

\[ Fig. 2 \text{ Comparison of } |x|, \text{ absr}(x), \text{ and } x^2 \]

2.1.6 Black-Box Sensitivity Evaluation. Obviously, incomplete
sensitivities can be obtained by linearizing the functional and
keeping all state based quantities unchanged. However, it might be
interesting to avoid any extra programming effort for the user.
This is a demand from industry where people are often not pro-
fessional enough or are black-box solver users. This is one of the
main interests of gradient free approaches such as genetic algo-
rithms. We describe a possible implementation of incomplete sen-
sitivities where a change in the functional does not imply any new
coding for the calculation of the gradient other than coding the
functional itself. We also take this opportunity to show how to
adapt an existing optimization platform to incomplete sensitivities.

2.1.7 Finite Difference Method. The easiest and most used
approach to compute sensitivities is to apply Taylor expansion and
to find each component of the gradient individually \((e^j\) denotes
variation along \(j\)th component)

\[ \frac{dJ}{dx_i} \approx \frac{1}{\epsilon} (J(x_i + \epsilon e^j, q(x_i + \epsilon e^j), W(x_i + \epsilon e^j))\]  

\[ - J(x_i, q(x_i), W(x_i)) \]  

(11)

Incomplete sensitivities approach suggests freezing the state vari-
able in the previous formula

\[ \frac{dJ}{dx_i} \sim \frac{1}{\epsilon} (J(x_i + \epsilon e^j, q(x_i + \epsilon e^j), W(x_i)) - J(x_i, q(x_i), W(x_i)) \]  

(12)

This is easy to do using finite differences in an existing optimiza-
tion procedure.

2.1.8 Complex Variable Method. The drawbacks of difference
formulas are well known (choice of the increment and difference
between two close quantities). These can be avoided when work-
ing with complex values [13,14] where

\[ \frac{dJ}{dx_i} = \text{Im}[J(x_i + i\epsilon, q(x_i + i\epsilon e^j), W(x_i + i\epsilon))] \]  

\[ \frac{1}{\epsilon} \]  

(13)

Here again the incomplete sensitivity method suggests the follow-

\[ \frac{dJ}{dx_i} \sim \frac{\text{Im}[J(x_i + i\epsilon, q(x_i + i\epsilon e^j), W(x_i))]}{\epsilon} \]  

(14)

where \(i = \sqrt{-1}\). In practice, this method only requires a redefinition
of all real variables of a computer program as complex. This is not
convenient if a black-box solver is used. But with incomplete
sensitivities, only the boundary integral calculations are involved.

2.1.9 Minimization Method. We briefly recall the minimiza-
tion method to show where incomplete sensitivities appear in
descent iterations. Our main interest goes to quasi-Newtonian meth-
ods such as BFGS coupled with a line search method [15]. The
approximate inverse of the Hessian of the functional is built using
successive gradient evaluations. Therefore, with incomplete sen-
sitivities one might expect not only a deviation in the gradient but
also in eigenvalues of the Hessian.

Consider the following general minimization problem [15] with
box constraints:

\[ \min f(x), \quad l \leq x \leq u \]  

\[ x \in \mathbb{R}^n \]  

(15)

A quasi-Newton method searches for the minimum of the follow-

\[ \min \left[ \frac{1}{2} d^T B d + g^T f + f(x) \right] \]  

(16)

The minimum occurs if \(Bd + g = 0\). Therefore at point \(x_c\), the
search direction is defined by \(d = -B^{-1}g\), where \(B\) is a positive
definite approximation of the Hessian and \(g\) is the gradient (in-
complete) at \(x_c\).

A line search is then used to find a new point \(x_n\)

\[ x_n = x_c + \lambda d \]  

\[ \lambda \in (0, 1] \]  

(17)

such that
\[
f(x_n) = f(x) + \alpha g^T d
\]
(18)

Finally, the optimal conditions
\[
|g_n(x_i)| \leq \epsilon, \quad l_i < x_i < u_i
\]
\[
g_n(x_i) < 0, \quad x_i = u_i
\]
\[
g_n(x_i) > 0, \quad x_i = l_i
\]
are checked, where \( \epsilon \) is a gradient tolerance. When optimality is not achieved, \( B \) is updated according to the BFGS [15] formula (starting from identity matrix)
\[
B \leftarrow B - \frac{B \delta \delta^T B}{\delta^T B \delta} + \frac{\delta x_n - x_c}{\gamma_n g_n g_n^T} \frac{\delta x_n - x_c}{\gamma_n g_n g_n^T}
\]
(20)

Another search direction is then computed to begin the next iteration.

If one denotes \( g_b = I_n + \epsilon_n \) and \( g_s = I_n + \epsilon_s \), where \( I \) is the incomplete sensitivity and \( E \) is the remaining part, Eq. (20) suggests that for functionals in the validity domain of incomplete sensitivities, one has equidistribution of the error \( E_n - \epsilon_n = O(1) \) and \( |E_n| - |\epsilon_n| \).

### 2.1.10 3D Flow Simulation

To have an efficient shape optimization for fluids, the optimization platform should be able to interact with various computational fluid dynamics (CFD) solvers. To achieve such adaptability, it is important to keep the interface free of constraints for a particular software. This is also one of the advantages of the incomplete sensitivity concept, as it lets the interface to be only surface based. FINE/TURBO, developed by Numeca, is an integrated software based on finite volume discretization for multiblock structured grids. The multiblock structured grids on the blades were prepared by AUTOGIRD developed by Numeca [16]. The physical model used in the solver was the Reynolds-averaged Navier–Stokes equations in rotating frames of reference coupled with various turbulence models and near-wall treatment for low-Reynolds modeling. The standard high Reynolds \( k-\epsilon \) turbulence model with extended wall functions could be chosen without any limitation [17–19].

The discrete schemes were second order in space [17] and first order in time with time marching to steady solutions. Mass flow rate, velocity direction, turbulence kinetic energy \( k \), and turbulent dissipation \( \epsilon \) were imposed at the inlet boundary, while at outlet boundary condition static pressure was prescribed. Finally, a periodic boundary condition was applied between two blades.

### 2.2 Rounding of Blade’s Leading Edge and Hub/Shroud Inlet Edges of Reverse Pump

Figure 3 illustrates modifications of the edges, which involves rounding of the shape blade and the sharp edges of the hub/shroud in the inlet of the reverse pump’s impeller. On the blade edge profile, the rounding \( r \) is taken to be equal to half of the blade thickness, and care is taken to ensure that the overall diameter is not altered. Therefore, there is a certain trade-off in selecting the rounding radius \( r \). The hub and shroud are subjected to rounding both at the outer and at the inner edges of the hub and shroud.

This modification could reduce the net flow separation loss component at the blades and impeller. However, this modification can also cause a rearrangement of inlet velocity triangle that can change the shock loss component. Loss reduction associated with the shock wave coming in contact with smoother geometric profiles and complex interaction in the clearance between volute and impeller (with rounded hub/shroud) can be expected.

### 3 Experimental Setup

A complete laboratory model of minihydropower plant was installed in University of Tehran [1], as shown in Fig. 4. The flow rate and head for the pump working as a turbine were generated in the experimental setup by several pumps.

When a pump works as a turbine, a control system is needed to automatically regulate the frequency. The classical governor used for standard turbines are expensive and not always recommended for small hydropower plants. Since these types of plants are being used more in isolated areas, an electronic load controller with ballast loads was built and used for keeping the frequency of the generator in these tests. A conventional synchronous generator...
was installed for producing electricity. For turbine shaft torque measuring, generator was changed to suspense state mode and using a scaled arm and several weights; the turbine shaft torque was measured. The flow rate was measured by the discharge law using various orifice plates for each test. Pressures were measured by some barometers. An industrial low specific speed centrifugal pump with a specific speed of 23.5 (m, m³/s) was selected for testing as a turbine with one original impeller and three modified impellers. This pump had maximum input turbine shaft power, maximum head, and maximum flow rate of 20 kW, 25 m, and 120 l/s, respectively. For the reverse pump testing, a feed pump, several pipes, an orifice, a generator, and ballast loads were selected and installed in the test rig. In the application of the reverse pump, it should be considered that if a generator is to be coupled directly, a nominal speed corresponding to one of the synchronous speeds (e.g., 750 rpm, 1000 rpm, 1500 rpm, or 3000 rpm) should be chosen. For induction generators, and also induction motors, the slip factor must be taken into account (the tested pump rotates at 1450 rpm in pump mode). In practice, synchronous generators are usually used. The reverse pump was tested in \( N_t = 1500 \) rpm.

After measuring all parameters, the reverse pump head, flow rate, output power, and efficiency were obtained. A first-order uncertainty analysis is performed using the constant odds combination method based on a 95% confidence level, as described by
The uncertainty of the head, flow rate, power, and efficiency are, respectively, 5.5%, 3.4%, 5.1%, and 5.5%.

4 Results

We considered a centrifugal pump in reverse rotation with a rotational speed of 1500 rpm, a flow rate of 126 m$^3$/h, and a total head rise of 38 m. The pump had seven blades with an inlet radius in the hub of 0.25 m. This pump was tested as a turbine in the test rig. The shape of impeller blades were optimized by the described optimization algorithm to reach higher efficiency in the rated point region. The cost function for optimization was

$$J = -\frac{T_r}{T_{r0}} + 0.05 \frac{|F_a - F_{a0}|}{F_{a0}} + 0.1 \frac{|F_R - F_{R0}|}{F_{R0}} + 0.001 \frac{|V_b - V_{b0}|}{V_{b0}}$$

The initial geometry was available at hub-span, midspan, and shroud-span. The mesh used for FINE/TURBO was structured, multi-block, and of the HHOH/O5H-type. This was an elliptic mesh with about 400,000 nodes. The computational domain and grid view in midspan are shown in Fig. 5. To check if the grid was too coarse, simulations were made with one impeller channel and two different grids. The first grid had about 150,000 cells for one impeller channel and the second one consisted of about 400,000 cells. The simulation results showed differences of less than 1% for efficiency and head. The cost function included not only the

<table>
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Fig. 7 Optimization for the second parametrization of a centrifugal pump blade. (a) Blade performance versus optimization iterations ($\eta/\eta_0$, $h/h_0$, $T_r/T_{r0}$). (b) Initial and final blades, hub-span, midspan, and shroud-span.
torque to maximize, but also a state constraint on hydrodynamic forces as well as a geometrical constraint on the blade volume.

Results showed that the torque was increased by 4.25% and the head by 1.97% for a hydraulic efficiency improvement of 2.2% (see Fig. 6 and Table 1).

For the second optimization with the same cost function, the torque was increased by 2.27% and the head by 1.08% for a hydraulic efficiency improvement of 1.17% (see Fig. 7 and Table 1).

The initial and final optimization results are shown in Table 1. The final designs were more robust than the original shape as the gradients of all constraints were reduced. The optimization process was reasonably fast and required about 17 iterations of the optimization algorithm and 26 functional evolutions. On a 3 GHz computer with a 4 Gbyte RAM, the flow analysis and the complete optimization took almost 23 h and 2 h, respectively.

In the next modification, the leading edges of blades and the inlet edges of the hub and shroud were rounding according to Fig. 3. Finally, all impellers were manufactured and tested in the test rig.

Figures 8 and 9 show the results of the experiments. In these figures, $\psi$, $\phi$, and $\pi$ are defined as

$$\psi = \frac{gH}{nD^2}, \quad \phi = \frac{Q}{nD^3}, \quad \pi = \frac{P}{\rho n^2 D^5}$$

Table 2 shows the changes in hydraulic parameters in flow rate of best efficiency point (BEP). Efficiency improvement occurs in all flow rates of part load and overload zones. The optimized impeller gives an increase of $-2.2\%$, $+9.4\%$, $+14.8\%$, and $+2.9\%$ for head, power, and efficiency, respectively. Table 3 shows the comparison between experimental and numerical optimization results. The experiment shows higher values for power, head, and efficiency. This confirms the numerical results and shows that the blade behavior was improved for a wider operating range. But the head is increased slightly more than numerical optimization data. In the optimized geometry, the inlet blade angle is bigger than that of the initial one.

Rounding of optimized impeller improved these values to $+5.5\%$, $11.5\%$, $36.1\%$, and $5.5\%$ for flow rate, head, power, and efficiency respectively. This modification could reduce net flow separation loss component at blades and impeller and also cause a rearrangement in the inlet velocity triangle and a change in the shock loss component.

\[\text{Fig. 8 Experimental results for head number and efficiency}\]

\[\text{Fig. 9 Experimental results for power number and efficiency}\]
As we saw, by doing easy modifications on the impeller of the reverse pump, its maximum efficiency was increased.

5 Conclusions

Using the gradient based optimization process on the radial turbomachinery blade design developed in the authors’ previous work [9], the blade shape of the impeller of a reverse pump was optimized to improve its maximum efficiency in rated point. The new design was experimentally tested for several operating points. The efficiency was improved in all measured points. Additional modification was the rounding of the blades profile in impeller inlet and hub/shroud inlet edges in turbine mode. Experimental results confirmed that the efficiency is improved in all measured points for this modification.

This study illustrated that just by impeller modification the efficiency of the pump in reverse operation can be improved.

Appendix: Sensitivities Improvement by Reduced Order Models

A middle path between full state equation linearization and incomplete sensitivities is the linearization low-order models to recover the neglected part in the gradient [9]. In other words, one would like to linearize the original state equation

\[ x_c \rightarrow q(x) \rightarrow W(q) \] (A1)

But the following form where the last normalization term is frozen. It must be easier to evaluate and linearize the low-complexity model \( \bar{W}(q) - W(q) \): \n
\[ x_c \rightarrow q \rightarrow \bar{W}(q) \left( \frac{W(q)}{\bar{W}(q)} \right) \] (A2)

where \( \bar{W} \) is the solution of a reduced order model. The last term is for the reduced order model to produce the same results. The incomplete gradient of \( J \) with respect to \( x_c \) can be improved by approximating the neglected part of the exact gradient by the linearized reduced order model:

\[ \frac{dJ}{dx_c} = \frac{\partial J}{\partial x_c} + \frac{\partial J}{\partial q} \frac{dq}{dx_c} + \frac{\partial J}{\partial \bar{W}} \frac{d\bar{W}}{dx_c} \left( \frac{W}{\bar{W}} \right) \] (A3)

One notices that \( \bar{W} \) is never used in state evaluation, but only \( \partial \bar{W}/\partial q \) in the sensitivity evaluation. Also, the reduced order model needs only to be locally valid and is linearized around the solution of the full model. The scaling term in Eq. (22) also helps in recovering the right level for the gradient [2].

The expression of the discrete gradient calculated with respect to an arbitrary normal to the shape \( n_j \) is

\[
\frac{dJ}{dn_j} = \left( \sum_{i=1}^{N_s} \frac{d}{dn_j} [T_i \cdot n_j \text{area}(e_i)] \right) \cdot \sigma
\]

\[= \sum_{i=1}^{N_s} \left( \frac{\partial T_i}{\partial n_j} \text{area}(e_i) + T_i \cdot \frac{\partial n_j}{\partial n_j} \text{area}(e_i) + [T_i \cdot \frac{\partial \text{area}(e_i)}{\partial n_j}] \right) \cdot \sigma \] (A4)

where \( n \) denotes the normal to the shape, \( \sigma \) is an arbitrary direction, \( dV_b \) is the surface increment, \( e_i \) is a discrete surface element, and \( J_b \) is the discrete cost function.

Following the idea of gradient approximation by neglecting the state contributions, it is assumed that the state in a given point does not depend on its value at the other points \( \partial T_i/\partial n_j = 0 \) for \( i \neq j \). Therefore the state term in the sensitivities can be evaluated in an arbitrary point on the shape using reduced order models. State linearization is given by

\[
\frac{\partial T}{\partial n} \cdot n = \frac{\partial}{\partial n} (p t - (v + v_n)(\nabla u_t + \nabla u^T)) \cdot n
\]

\[= \frac{\partial p}{\partial n} \cdot n - \frac{\partial}{\partial n} [(v + v_n)(\nabla u_t + \nabla u^T)) \cdot n \] (A5)

The selection of reduced order models depends on the application of interest. In this work, boundary layer theory can be used by considering a local reference frame in a point of the shape with \( y \) indicating the direction normal to the wall and \( y_w \) denoting the wall location.

For sensitivity of the pressure and viscous shear with respect to variations in the shape in the normal direction, the terms to be evaluated in Eq. (24) are equivalent to the following in the local referential:

\[
\frac{\partial p}{\partial y_w}, \quad \frac{\partial}{\partial y_w} \left( (v + v_n) \frac{\partial u}{\partial y} \right) \] (A6)

where \( u \) denotes the tangential velocity. In boundary layer theory, the pressure gradient normal to the wall is zero. So in this case if the shape variations are normal to the wall, pressure variation vanishes with respect to the small variation in \( y_w \):

\[
\frac{\partial p}{\partial y_w} = 0
\] (A7)

For viscous effects, wall laws can be used as reduced order models. These models replace the no-slip boundary conditions by a relation between the variables and their derivatives for the tangential component of the velocity together with a nonpenetration condition

\[
\frac{\bar{u}}{u_v} = f(y^*(u_v)) \] (A8)

where \( u_v = \sqrt{\tau_y/\rho} \), the local friction velocity, is computed using the flow state provided by the Navier–Stokes solver, and the local Reynolds number is defined as \( y^* = (y - y_w)u_v/\nu \).

Using boundary layer assumption and shear conservation along the normal direction, the sensitivity term to be evaluated is

\[
\frac{\partial}{\partial y_w} \left( (v + v_n) \frac{\partial u}{\partial y} \right) = \frac{\partial \tau_w}{\partial y_w} = 2u_w \frac{\partial u_w}{\partial y_w} \] (A9)

Now one can use the velocity gradient from the flow solver.
\[ \frac{\partial \tilde{u}}{\partial y_w} = \nabla u \cdot n(y_w) = \frac{\partial u}{\partial y_w} f(u_t) + u_t \frac{\partial f(u_t)}{\partial y_w} \]  
(A10)

This eventually gives the expression for shear sensitivity with respect to the normal deformation to the wall

\[ 2u_t \frac{\partial u_t}{\partial y_w} = 2u_t^2 \left( \nabla u \cdot n(y_w) - u_t \frac{\partial f}{\partial y_w} \right) / u \]  
(A11)

As mentioned, this is calculated around the “Navier–Stokes” solution (i.e., \( u_t, \nabla u, \) and \( u \) come from the flow solver).

Using, for instance, the universal log-low \( f(u_t) = \ln(y^+) / \kappa + B \) with \( \kappa = 0.41 \) and \( B = 5.36 \):

\[ \frac{\partial f}{\partial y_w} = \frac{-1}{\kappa(y - y_w)} \]  
(A12)

This approach provides information on the state sensitivity with respect to the shape variations along the normal direction to the wall. In cases where incomplete sensitivities have both tangential and normal components to the shape, this contribution shall be added only to the normal component. If one would like to be efficient with incomplete sensitivities, it is wise to choose a shape variation parameterization normal to the shape.

References