Incomplete sensitivities for 3D radial turbomachinery blade optimization

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Abstract

We are interested in optimal design of 3D complex geometries, such as radial turbomachines, in large control space. The calculation of the gradient of the cost function is a key point when a gradient based method is used. Finite difference method has a complexity proportional to the size of the control space and the adjoint method requires important extra coding. We propose to consider the incomplete sensitivities method for optimal design of radial turbomachinery blades. The central point of the paper is how to adapt some formulations in radial turbomachinery to the validity domain of incomplete sensitivities. Also, we discuss on how to improve the accuracy of incomplete sensitivities using reduced order models based on physical assumptions. Fine/Turbo flow solver is coupled with gradient based optimization algorithms based on CAD-connected frameworks. Newton methods together with incomplete expressions of gradients are used. The approach is validated through optimization of centrifugal pumps. Finally the results are considered and discussed.

1. Introduction

One major limitation of gradient based methods for optimization problems is the computation of the gradient of the cost function. Adjoint method based on control theory [1,2] can reduce the cost of this calculation at the cost of developing a dedicated solver for the adjoint variable. This method is particularly efficient when the dimension of the control space is large. This difficulty is more penalizing when industrial black-box flow solvers are used for the state, which is nowadays a systematic demand. Finite differences permit to get the sensitivity of black-box solvers, but then the cost of the evaluation is proportional to the size of the control space. Incomplete sensitivity method is a possible choice for calculating approximate gradient at practically no cost. In addition, incomplete sensitivities can be computed for individual constraint at no extra cost. Providing individual sensitivities for constraints is useful in robust optimization as one would like to see the sensitivity of the final design for small perturbation of control parameters for the different functionals involved. However, incomplete sensitivities have a limited validity domain: the cost function must be defined over the shape, or part of it, and must involve product of state by geometry functionals. Functionals based, for instance, on aerodynamic coefficients enter this class. But, we will see that functionals involved in radial blade design do not belong to this validity domain. This paper is to propose a suitable reformulation of the problem.

The paper is organized as follow. The optimization problems with various shape parameterizations for radial turbomachinery blade geometries are presented. Fine/Turbo flow solver developed by Numeca [3] is briefly described. This solver is based on the solution of the Reynolds-averaged Navier–Stokes equations with the standard $k$–$\epsilon$ turbulence model and wall function for near-wall treatment [3]. Quasi-Newton minimization algorithms and, especially, the approximate Hessian inverse BFGS update formula [4] are recalled. This is important as it brings a posteriori validation of incomplete sensitivities showing...
that the approximate inverse of the Hessian of the functional based on incomplete sensitivities is correct [1]. We will also discuss how to use low complexity models based on boundary layer theory and wall functions to improve these sensitivities. The implementation of these can be either by direct linearization leading to boundary integrals for the sensitivity or using approaches not requiring any extra coding for the optimization part from the end-user. For this former approach, the natural choice to get sensitivity is finite difference. On the other hand, the complex variable method has the advantage of not being sensible to the choice of the increment in finite differences as well as errors introduced from differencing two close quantities [5, 6]. Finally, the application of this ensemble is shown to the realistic configurations of centrifugal pumps. These pumps have been previously designed to be efficient from the point of view of energy consumption. Therefore, this is a good evaluation of incomplete sensitivities near optimal functioning points. Also, because the fluid is water or oil and not air even one percent efficiency improvement is of utmost importance.

2. Shape optimization

The general form of the shape optimization problem can be written as [7]:

$$\begin{align*}
\min J(x_c, W(x_c), \nabla_{x_c} W(x_c)) \\
g(x_c) = 0 \\
g_1(x_c) \leq 0, \quad g_2(W(x_c)) \leq 0
\end{align*}$$

(1)

where $x_c$ is the control variable for the shape, $W$ the flow variable, $S$ the state equation, $g_1$ the geometrical constraints, $g_2$ the state constraints and $J$ the cost function that should be minimize. The optimization algorithm can be summarized as:

0. Provide initial shape parameterization $x_c^{0}$.
1. For $k = 2, 3, \ldots, k_{\text{max}}$ Do:
   1.1 Compute the flow state: $W(x_c^{k})$.
   1.2 Compute the cost function: $J(x_c^{k})$.
   1.3 Compute the incomplete sensitivity of the cost function:
   \[ \frac{\partial J(x_c^{k})}{\partial x_c} \]
   1.4 If $\left( \frac{\partial J(x_c^{k})}{\partial x_c} \right)^2 < \varepsilon$ or $J(x_c^{k}) < \varepsilon$: STOP.
   1.5 Compute $x_c^{k+1}$ minimizing $J$ using the incomplete gradient and the approximate inverse of Hessian by BGFS and evaluate $W(x_c^{k+1})$ and $J(x_c^{k+1})$ when it is necessary.

2.1. Shape parameterization

Several parameterizations are possible to describe aerodynamic or hydrodynamic shapes. In radial turbomachinery one can be considered as the spanwise blade angle distribution from leading to trailing edges. The performance of a radial turbomachine (i.e. centrifugal pump) is intensely influenced by these blade angles [8]. Previous results have shown that hydraulic efficiency is not sensitive to small perturbations of blade thicknesses [8]. Hence, in current optimizations initial blade thicknesses are frozen. Also, the meridian plan of the hub and shroud and the outlet diameter (in centrifugal pump) are fixed. The optimization is performed in two steps.

2.1.1. Primal optimization

For a radial blade, the camber lines in hub-span, mid-span and shroud-span are linked through the following relation:

$$d \theta(R) = \theta(R) - \theta_{\text{initial}}(R) = c_1(R - R_1) + c_2(R - R_1)^2 + c_3(R - R_1)^3$$

(2)

where $\theta$ is the tangential angle, $R$ the blade radius, $R_1$ the radius of the blade leading edge in hub-span (in centrifugal pump). The profile is fixed in $R_1$. The coefficients $c_1$, $c_2$ and $c_3$ are control parameters (Fig. 1a).

2.1.2. Final optimization

The optimal shape from primal parameterization is considered as initial guess for second level parameterization.
The camber lines of mid-span and shroud-span are linked to the blade camber line in hub-span through:

$$d\varphi = \frac{c_{2i-1}(s) + c_{2i-2}(s)^2 + c_{2i-3}(s)^3}{s}$$

where \(i = 1, 2, m = \int_{\mathcal{S}} d\mathcal{M}, d\mathcal{M} = \sqrt{dR^2 + dz^2}, \) in \(m - \theta\) conformal plan. \(s = \sqrt{m^2 + \theta^2} .\) Here the blade camber lines in mid-span and shroud-span are rotated around trailing edge which is fixed with respect to the hub-span camber line of the blade (Fig. 1b).

### 2.2. Sensitivity and incomplete sensitivity

The gradient of a cost function \(J(x_c, q(x_c), W(q(x_c))))\), function of shape control parameters \(x_c\), geometric entities \(q(x_c)\) (normal, volume, surface, ...) and state variables \(W\) can be derived using chain rule:

$$\frac{dJ}{dx_c} = \frac{\partial J}{\partial x} \frac{\partial q}{\partial x} + \frac{\partial J}{\partial W} \frac{\partial W}{\partial x}$$

where \(f\) and \(g\) are functions involving geometric quantities and state quantities respectively. Since the gradient approximation is based on the flow information on the shape, the cost function should be expressed as a function of the aerodynamic coefficients, or more generally [1]

$$J = \int \left[ f(x, q(x))g(W(q(x))) \right] d\Gamma$$

The dominant part in the gradient comes from geometrical quantities sensitivities and not from state linearization [1]. More precisely, the last term in gradient expression can be neglected

$$\frac{dJ}{dx_c} \approx \frac{\partial J}{\partial x} \frac{\partial q}{\partial x} + \frac{\partial J}{\partial W} \frac{\partial W}{\partial x}$$

This gradient approximation avoids the evaluation of an adjoint state and decreases the computational cost. Typical functionals in this class are aerodynamic or hydrodynamic forces on a shape along an arbitrary direction as

$$T_{r} = \left( \int_{\Gamma} [T \cdot n] d\Gamma \right) \cdot \sigma$$

where \(T = -pl + (v + v_t)(\nabla u_x + \nabla u_t)\). Here, \(n\) is the normal to the shape, \(\sigma\) an arbitrary direction and \(T\) the Newtonian stress tensor.

### 2.3. Cost function and its reformulation

A given optimization problem is not necessary suitable for incomplete sensitivity in its validity domain. In airfoil optimization it is easy to define the cost function as the drag and lift [1], Stanciu et al. [9] adapted incomplete sensitivities to axial turbomachinery. Our aim here is to adapt the approach to the design of radial turbomachines and centrifugal pumps. A centrifugal pump is designed for a given design point such as a known specific speed: \(N_s = \frac{N}{\sqrt{h \cdot q}}\) where \(h\) is the head (m), \(q\) the flow rate (m³/s) and \(N\) the rotational speed (rpm). The pump maximum hydraulic efficiency is defined as

$$\eta_h = \frac{q \Delta p_r}{T_r \omega}$$

where \(T_r\) (N m) is the axial torque from the impeller to the fluid, \(\omega = \frac{d\theta}{dt}\) and \(\Delta p_r\) (N/m²) total pressure difference.

To use incomplete sensitivities, the cost function must be based on information over the shape (or part of it). In Eq. (8) reducing the torque improves efficiency. One consider therefore the functional \(J = -\frac{T_r}{\omega}\) where:

$$T_r = \int_{\Gamma} [T \cdot n] d\Gamma, T = -pl + (v + v_t)(\nabla u_x + \nabla u_t)$$

But one needs to improve the hydraulic efficiency of the pump at constant design point (constant specific speed). So looking for higher hydraulic efficiency should be at given head (total pressure different between outlet and inlet) and also flow rate. The flow can be imposed through boundary conditions and is constant in optimization process. Head (pressure difference) can be added as a state constraint by penalty in the cost function

$$J = \frac{T_r}{\omega} + \frac{\eta_h}{h_0}$$

Unfortunately, this new term does not enter to incomplete sensitivity validity domain as it is defined away from the shape and also does not include any geometric quantity. Using the momentum theorem in rotating frame of reference (Fig. 2) one can write [10]

$$\vec{F} = \int_{\Gamma_i} d\mathcal{M}_{i1} \vec{W}_1 - \int_{\Gamma_i} d\mathcal{M}_{i2} \vec{W}_2 + \int_{\Gamma_i} (-\vec{n}_i \rho_1 ds_1)$$

$$+ \int_{\Gamma_i} (-\vec{n}_2 \rho_2 ds_2) + \int_{\Gamma_i} (-\vec{\tau}_1 ds_1) + \int_{\Gamma_i} (-\vec{\tau}_2 ds_2)$$

$$- \int_{\Gamma_i} (\vec{\rho} \omega \times \vec{W}) d\Gamma - \int_{\Gamma_i} \rho \vec{\omega} \times (\vec{\omega} \times \vec{W}) d\Gamma$$

where \(d\mathcal{M}\) is the mass flow rate, \(W\) the relative velocity, \(n\) the normal on the shape, \(\tau\) the viscous stress tensor, \(\rho\) the density and \(C_v\) the control volume. Indices 1,2 and \(w\) respectively denote inlet, outlet and rotor wall. \(F\) is the hydrodynamic force on the rotor.

Terms 5 and 6 in the right-hand side of Eq. (10) can be ignored by neglecting viscous effects at inlet and outlet regions.

The momentum theorem in \(\vec{i}_2\) direction (Fig. 2) is as

$$F_{\vec{i}_2} = \int_{\Gamma} d\mathcal{M}_{i1} \vec{W}_1 - \int_{\Gamma} d\mathcal{M}_{i2} \vec{W}_2$$

$$+ \int_{\Gamma} (-\vec{i}_2 \rho_1 \vec{n}_i ds_1) + \int_{\Gamma} (-\vec{i}_2 \rho_2 \vec{n}_i ds_2)$$

$$- \int_{\Gamma} 2\vec{\rho} \vec{i}_2 \vec{\omega} \times \vec{W} d\Gamma - \int_{\Gamma} \rho \vec{i}_2 \vec{\omega} \times (\vec{\omega} \times \vec{W}) d\Gamma$$

(11)
Summarizing, it can be as

\[ F_a = \rho s_1 W_{a1}^2 + p_1 s_1 \rightarrow p_1 + \rho \frac{W_{a1}^2}{2} = \frac{F_a}{s_1} - \rho \frac{W_{a1}^2}{2} \]

\[ \rightarrow p_{t1} = \frac{F_a}{s_1} - \rho \frac{W_{a1}^2}{2} \tag{12} \]

Eq. (12) is a proper passage between axial forces on the rotor from the fluid and the inlet total pressure. \( W_{a1} = \frac{a_1}{s_1} \) is constant in optimization process. Therefore:

\[ p_{t1} = \alpha_1 F_a + C_1 \tag{13} \]

Considering the momentum theorem in \( \vec{T}_3 \) direction (Fig. 3):

\[ \begin{align*}
F_R &= \vec{T}_3 
F \\
&= \int_{s_1} dM_{i1}\vec{T}_3 \vec{W}_1 - \int_{s_2} dM_{i2}\vec{T}_3 \vec{W}_2 \\
&+ \int_{s_1} (-\vec{T}_1 \vec{n}_1 p_1 ds_1) + \int_{s_2} (-\vec{T}_1 \vec{n}_2 p_2 ds_2) \\
&- \int_{C_r} 2\rho\vec{T}_3 \vec{\omega} \times \vec{W} dv - \int_{C_r} \rho\vec{T}_3 \vec{\omega} \times (\vec{\omega} \times \vec{W}) dv \tag{14} \end{align*} \]

Summarizing, it can be as

\[ p_{t2} = -\frac{F_R}{s_2} - \rho \frac{W_{R2}^2}{2} + \frac{3\rho \omega^2}{s_2} \int_{C_r} R dv \tag{15} \]

Similarly, \( W_{a2} = \frac{a_2}{s_2} \) is constant in optimization process. In the right hand side of Eq. (15), term 3 can be expressed as

\[ \int_{C_r} R dv = \int_{C_r} R dv \bigg|_{\text{implier}} - m \int_{C_r} R dv \bigg|_{\text{blade}} \]

\[ \int_{C_r} R dv \bigg|_{\text{blade}} = R V_b \tag{16} \]

Assuming that variation of \( R \) in optimization process is small, one can be derived as

\[ p_{t2} = -\alpha_2 F_R - \alpha_3 V_b + C_2 \tag{17} \]

Eq. (17) illustrates the relation between the rotor outlet pressure and the radial forces and the blade volume. Using Eqs. (13) and (17) the total pressure difference can be expressed as

\[ h = p_{t2} - p_{t1} = -\alpha_1 F_a - \alpha_2 F_R - \alpha_3 V_b + C \tag{18} \]
Eventually, the cost function for efficiency improvement and constant head can be reformulated from (9) to
\[
J = \frac{T_o}{\dot{r}_o} + \frac{2}{F_o} \frac{|F_a - F_{o0}| + \beta |F_R - F_{R0}|}{F_{o0}} + \gamma \frac{|V_h - V_{h0}|}{V_{h0}}
\]
(19)
which enters incomplete sensitivity validity domain. Indeed, the cost function is the rotor axial torque with constraints on hydrodynamic axial and radial forces and geometrical constraint on the blade volume. Hydraulic efficiency will be improved by decreasing torque and at the same time keeping approximately unchanged \(F_a\), \(F_R\) and \(V_h\), and therefore the head. This is an example of how to reformulate the functional to apply incomplete sensitivities in cases the original functional is not suitable.

2.4. Sensitivity improvement by reduced order models

A middle path between full state equation linearization and incomplete sensitivities is linearizing low-order models to recover the neglected part in the gradient [9]. In other words, one would like to linearize not the original state equation
\[
x_c \rightarrow q(x) \rightarrow W(q)
\]
(20)
But the following form where the last normalization term is frozen. The low-complexity model \(W(q) \sim W(q)\) has to be easier to evaluate and linearize:
\[
x_c \rightarrow q \rightarrow \bar{W}(q) \left( \frac{W(q)}{W(q)} \right)
\]
(21)
where \(\bar{W}\) is the solution of a reduced order model. The last term is for the reduced order model to produce the same results than the full model. The incomplete gradient of \(J\) with respect to \(x_c\) can be improved by approximating the neglected part of the exact gradient by the linearized reduced order model:
\[
\frac{\partial J}{\partial x_c} \approx \frac{\partial \bar{J}}{\partial x_c} + \frac{\partial \bar{J}}{\partial q} \frac{\partial q}{\partial x_c} + \frac{\partial \bar{W}}{\partial q} \frac{\partial q}{\partial x_c} \left( \frac{W}{\bar{W}} \right) \frac{\partial q}{\partial x_c} \left( \frac{W}{\bar{W}} \right)
\]
(22)
One notices that \(\bar{W}\) is never used in state evaluation but only \(\frac{\partial W}{\partial q}\) in the sensitivity evaluation. Also, the reduced order model needs only be locally valid and is linearized around the solution of the full model. The scaling term in Eq. (22) also helps recovering the right level for the gradient [2].

The expression of the discrete gradient calculated with respect to an arbitrary normal to the shape \(n_j\) is
\[
\frac{\partial J}{\partial n_j} = \sum_{i}^{N_j} \frac{d}{dn_j} [T_i \cdot n_j] \text{Area}(e_i) \cdot \sigma
\]
\[
= \left\{ \sum_{i}^{N_j} \left[ \frac{\partial T_i}{\partial n_j} \cdot n_i \right] \text{Area}(e_i) + \left[ T_i \cdot \frac{\partial n_i}{\partial n_j} \right] \text{Area}(e_i) \right\} \cdot \sigma
\]
\[
+ \left[ T_i \cdot n_j \right] \frac{\partial \text{Area}(e_i)}{\partial n_j} \cdot \sigma \quad \text{where } n \text{ denotes the normal to the shape, } \sigma \text{ an arbitrary direction, } d\Gamma_h \text{ the surface increment, } e_i \text{ a discrete surface element and } J_r \text{ the discrete cost function.}
\]
Following the idea of gradient approximation by neglecting the state contributions, it is assumed that the state in a given point does not depend on its value at the other points (\(\frac{\partial q}{\partial n_i} = 0 \text{ for } i \neq j\)). Therefore the state term in the sensitivities can be evaluated in an arbitrary point on the shape using reduced order models. State linearization is given by
\[
\frac{\partial J}{\partial n_j} \cdot n = \frac{\partial}{\partial n} [pl - (v + v_i)(\nabla u + \nabla u^T)] \cdot n
\]
\[
= \frac{\partial p}{\partial n} \cdot n + \frac{\partial}{\partial n} [(v + v_i)(\nabla u + \nabla u^T)] \cdot n
\]
(24)
The selection of reduced order models depends on the application of interest. In this work boundary layer theory can be used by considering a local reference frame in a point of the shape with \(y\) indicating the direction normal to the wall and \(y_w\) denoting the wall location.

For sensitivity of the pressure and viscous shear with respect to variations in the shape in the normal direction, the terms to be evaluated in Eq. (24) are equivalent to the following ones in the local referential:
\[
\frac{\partial p}{\partial y_w} = - \frac{\partial p}{\partial y} = 0
\]
(26)
For viscous effects, wall laws can be used as reduced order models. These models replace the no-slip boundary conditions by a relation between the variables and their derivates for the tangential component of the velocity together with a non penetration condition:
\[
\frac{\partial}{\partial y} = f(y^+ (u_i))
\]
(27)
where \(u_i = \sqrt{\tau_n}\) the local friction velocity is computed using the flow state provided by the Navier–Stokes solver and the local Reynolds number is defined as \(y^+ = \frac{u_i y_w}{v}\).
Using boundary layer assumption and shear conservation along normal direction, the sensitivity term to be evaluated
\[
\frac{\partial}{\partial y_w} (v + v_i) \frac{\partial u_i}{\partial y} = \frac{\partial \tau_n}{\partial y_w} = 2u_i \frac{\partial u_i}{\partial y_w}
\]
(28)
Now one can use velocity gradient from the flow solver
\[
\frac{\partial \tilde{u}}{\partial y_w} = \nabla u \cdot n(y_w) = \frac{\partial u_i}{\partial y_w} f(u_i) + u_i \frac{\partial f(u_i)}{\partial y_w}
\]
(29)
This eventually gives the expression for shear sensitivity with respect to the normal deformation to the wall.
\[ 2u_t \frac{\partial u_t}{\partial y_w} = 2u_t^3 \left( \nabla u \cdot n(y_w) - u_t \frac{\partial f}{\partial y_w} \right) / u \]  

(30)

As mentioned, this is calculated around the ‘Navier–Stokes’ solution (i.e. \( u_t, \nabla u \) and \( u \) come from the flow solver).

Using, for instance, the universal log-low \( f(u_t) = \frac{\text{im}(e^0)}{n} + B \) with \( \kappa = 0.41 \) and \( B = 5.36 \)

\[ \frac{\partial f}{\partial y_w} = \frac{-1}{\kappa(y - y_w)} \]  

(31)

This approach provides information on the state sensitivity with respect to the shape variations along the normal direction to the wall. In cases incomplete sensitivity has both tangential and normal components to the shape this contribution shall be added to the normal component. If one would like to be efficient with incomplete sensitivities, it is wise to choose shape variations parameterization normal to the shape. Indeed, this reduces the sensitivity of the optimization to miscalculation of sensitivities making the optimization more robust.

2.5. Black-box sensitivity evaluation

Obviously, incomplete sensitivities can be obtained by linearizing the functional and keeping unchanged all state based quantities. However, it might be interesting to avoid any extra programming effort for the user. This is a systematical demand from industry where people are often not specialized or are black-box solver users. This is one of main interest of gradient free approaches such as genetic algorithms. We describe a possible implementation of incomplete sensitivities where a change in the functional does not imply any new coding for the calculation of the gradient other than coding the functional itself. We also take this opportunity to show how to adapt existing optimization platform to incomplete sensitivities.

2.5.1. Finite difference method

The easiest and most used approach to compute sensitivities is to apply Taylor expansion and find individually each component of the gradient (\( e^0 \) denotes variation along \( j \)th component)

\[ \frac{\text{d}J}{\text{d}x_c} e^l = \frac{1}{\epsilon} \left( J(x_c + \epsilon e^l, q(x_c + \epsilon e^l), W(x_c + \epsilon e^l)) - J(x_c, q(x_c), W(x_c)) \right) \]  

(32)

Incomplete sensitivities approach suggests freezing the state variable in previous formula

\[ \frac{\text{d}J}{\text{d}x_c} e^l \sim \frac{1}{\epsilon} \left( J(x_c + \epsilon e^l, q(x_c + \epsilon e^l), W(x_c)) - J(x_c, q(x_c), W(x_c)) \right) \]  

(33)

This is easy to do in an existing optimization procedure using finite differences. Indeed, it is sufficient reduce the required accuracy when calculating perturbed states starting from the state obtained for the shape \( x_c \).

2.5.2. Complex variable method

The drawbacks of difference formula are well known (choice of the increment and difference between two close quantities). These can be avoided working with complex values \([5,6]\) where:

\[ \frac{\text{d}J}{\text{d}x_c} \sim \frac{\text{Im}[J(x_c + iz, q(x_c + ize^0), W(x_c + iz) - J(x_c, q(x_c), W(x_c))]}{\epsilon} \]  

(34)

Here again incomplete sensitivities method suggests the following approximation:

\[ \frac{\text{d}J}{\text{d}x_c} \sim \frac{\text{Im}[J(x_c + iz, q(x_c + ize^0), W(x_c))]}{\epsilon} \]  

(35)

where \( i = \sqrt{-1} \). In practice, this method only requires a redefinition of all real variables of a computer program as complex. This is not convenient if a black-box solver is used. But, with incomplete sensitivities this is treated separately as only involves a boundary integral calculation user-routine.

2.6. Minimization method

We briefly recall the minimization method to show where incomplete sensitivities appear in descent iterations. Our main interest goes to quasi-Newton methods such as BFGS coupled with a line search method \([4]\). The approximate inverse of Hessian of the functional is built using successive gradient evaluations. Therefore with incomplete sensitivities one might expect not only a deviation in the gradient but also in eigenvalues of the Hessian. In other words, if the BFGS formula gives satisfaction one might conclude to either an equidistribution of error in successive gradients or to the fact that the errors in incomplete sensitivities are negligible during optimization.

Consider the following general minimization problem \([4]\) with box constraints:

\[ \{ \min f(x), \quad l \leq x \leq u \} \]  

(36)

A quasi-Newton method searches for the minimum of the following function in each iteration:

\[ \min_{x} \left[ \frac{1}{2} d^T B d + g_c^T + f(x) \right] \]  

(37)

The minimum occurs if \( Bd + g_c = 0 \). Therefore at point \( x_c \), the search direction is defined by \( d = -B^{-1}g_c \), where \( B \) is a positive definite approximation of the Hessian and \( g_c \) is the gradient (incomplete) at \( x_c \).

A line search is then used to find a new point \( x_n \)

\[ \{ x_n = x_c + \lambda d, \quad \lambda \in (0, 1) \} \]  

(38)

such that
\[
\begin{align*}
  f(x_o) &= f(x_c) + \alpha g^T d \\
  \alpha &\in (0, 0.5)
\end{align*}
\]

Finally, the optimality conditions:
\[
\begin{align*}
  |g_n(x_i)| &\leq \varepsilon, \quad l_i < x_i < u_i \\
  g_n(x_i) &< 0, \quad x_i = l_i \\
  g_n(x_i) &> 0, \quad x_i = l_i
\end{align*}
\]

are checked, where \(\varepsilon\) is a gradient tolerance. When optimality is not achieved, \(B\) is updated according to the BFGS [4] formula (starting from identity matrix):
\[
B \leftarrow B - \frac{B\delta \delta^T B}{\delta^T B \delta} + \frac{yy^T}{y^T \delta}, \quad \delta = x_n - x_c \\
y = g_n - g_c
\]

Another search direction is then computed to begin the next iteration.

If one denotes \(g_c = I_c + E_c\) and \(g_n = I_n + E_n\), with \(I\) the incomplete sensitivity and \(E\) the remaining part. Formula (41) suggests that for functionals in the validity domain of incomplete sensitivities one has equidistribution of the error: \(E_c \cdot E_n = O(1)\) and \(|E_c| \sim |E_n|\).

### 2.7. 3D flow simulation

Nowadays CFD is an industrial topic. In shape optimization for fluids, to be efficient, the optimization platform should be able to interface with various CFD solvers. To achieve such adaptability it is important the interface to remain free of constraints related to a particular software. This is also one advantage of incomplete sensitivities concept as it permits for the interface to be only surface based. Fine/Turbo developed by Numeca, is an integrated software based on finite volume discretization for multi-block structured grids. The multi-block structured grids on the blades have been prepared by AutoGrid5 developed by Numeca [11]. The physical model used in the solver is the Reynolds-averaged Navier–Stokes equations in rotating frames of reference coupled with various turbulence models and near-wall treatment for low-Reynolds modeling. Without limitation, it may be chosen the standard high Reynolds \(k-\varepsilon\) turbulence model with extended wall functions [3,12,13].

The discrete schemes are second order [3] in space and first order in time with time marching to steady solutions. Mass flow rate, velocity direction, turbulence kinetic energy \(k\) and turbulent dissipation \(\varepsilon\) are imposed at inlet boundary while at outlet boundary condition, static pressure is prescribed. Finally, periodic boundary condition is applied between two blades.

### 3. Results

The aim is to numerically validate the incomplete sensitivity concept and also to see the advantage of using low complexity models in sensitivities for applications in radial turbomachinery. A centrifugal pump is considered with the following characteristics:

- rotational speed: 1450 rpm
- inlet radius of impeller at hub: 0.25 m
- flow rate: 90 m\(^3\)/h
- inlet radius of impeller at shroud: 0.63
- total impeller head rise: 24.5 m
- outlet radius of the impeller: 0.269
- number of blades: 8

The initial geometry is available at hub-span, mid-span and shroud-span. The mesh used for Fine/Turbo is structured and multi-block. This is an elliptic mesh about 600,000 nodes. The computational domain and the \(B-B\) grid view in hub-span are shown in Fig. 3.

The aim is to maximize hydraulic efficiency in unchanged design point. We consider two cost functions:

![Blade performance vs. optimization iterations](image1)

(a) Blade performance vs. optimization iterations (\(\eta_h/\eta_{ho}, h/h_0, T/T_{ho}\))

![Initial and final blades](image2)

(b) Initial and final blades, hub-span, mid-span and shroud-span

![Optimization for the first parameterization of a centrifugal pump blade](image3)

Fig. 4. Optimization for the first parameterization of a centrifugal pump blade: \(J = T_s/T_{so}\): (a) Blade performance vs. optimization iterations (\(\eta_h/\eta_{ho}, h/h_0, T_s/T_{ho}\)). (b) Initial and final blades, hub-span, mid-span and shroud-span.
\[ J = \frac{T_r}{T_{r0}} \] and
\[ J = \frac{T_r}{T_{r0}} + \alpha \left( \frac{|F_a - F_{a0}|}{F_{a0}} \right) + \beta \left( \frac{|F_b - F_{b0}|}{F_{b0}} \right) + \gamma \left( \frac{|V_b - V_{b0}|}{V_{b0}} \right) \]

Both levels of parameterization introduced in Section 2, are checked. The first parameterization moves the total blade according to the hub-span camber line, in \( \theta \) direction using Eq. (2). The geometrical constraint is to keep the leading edge frozen along hub-span. The gradient is obtained using incomplete sensitivities and improved by reduced models. The second parameterization deforms the previous optimized geometry. The mid-span and shroud-span camber lines rotate around trailing edge according to Eq. (3) to keep the hub-span fixed.

The results are shown in Figs. 4–7 and Tables 1–3. In the first optimization, by primal parameterization based on incomplete sensitivities, the torque decreased by 16.4\%, the head by 15.5\% and hydraulic efficiency increased by 1.1\% (Fig. 5 and Table 1). The blade angle from leading to trailing edges, increased and consequently the blade length. For the second parameterization and the same cost function, the torque decreased by another 5.2\%, the head decreased by 2.1\% and hydraulic efficiency increased by 3.2\% (Fig. 6 and Table 1). Using upgraded incomplete sensitivities for optimization, the number of optimization iterations reduces from 17 (27 evaluations of \( J \)) to 13 (23 evaluations of \( J \)) in step 1 and from 17 (30 evaluations of \( J \)) to 14 (25 evaluations of \( J \)) in step 2 (Table 2).

Most important, the final shapes were the same with incomplete sensitivities and after their completion with reduced order models. In addition, one verifies that this shape is indeed an optimum as the full gradient computed by finite differences is small (Table 1). This shows that incomplete sensitivities are enough accurate to provide suitable descent directions for the quasi-Newton method to lead to the same final shape. However, using reduced order models incomplete gradient are more accurate leading to faster convergences. Tables 1 and 3 show that the
neglected parts in incomplete sensitivities are indeed negligible. It also shows that the reduced order models improve the approximate gradients. Linearizing reduced order models adds physical information to the geometric gradient.

In previous optimizations, the head decreases and therefore the specific speed changes. But pump optimal design is required at fixed specific speed. Therefore the cost function should include not only the torque to minimize but also a constraint on hydrodynamic forces as well as a constraint on the blade volume as specified in Eq. (15). Results show that the torque deceased by 4.1%, the head by 2% and the hydraulic efficiency improved by 2.2% (Fig. 7 and Table 3) in primal parameterization. For the second parameterization with the same cost function, the torque reduction is 2.3%, the head reduction 1.1% and the hydraulic efficiency improvement 1.2% (Fig. 8 and Table 3).

Tables 1 and 3 show the gradients of the cost function. The final design appears being also more robust than the original shape as the gradients of all constraints have been reduced. The maximum missing part of the incomplete gradient for the first cost function is 16% of the total gradient and 8% after updated by reduced order modeling.
second cost function the maximum deviation is 10% (Tables 1 and 3).

The optimization process is reasonably fast and requires about 15 iterations of the optimization algorithm and 25 functional evolutions. On a 3 GHz computer with 2GB of RAM the flow analysis takes about 2 h and the complete optimization another 2 h. This shows that the incomplete sensitivity method can make the simulation and design to have the same cost for a realistic application.

4. Conclusions

The complexity in gradient evaluation is a key element in gradient-based optimization. Incomplete sensitivity concept has been used to decrease the cost of sensitivity evaluation and also to avoid an adjoint state solver development unsuitable in black-box environments. Since incomplete sensitivities are limited to cost functions based on shape information and involving geometry by state functions product, functional reformulations has been performed to adapt the original problem. Since improving hydraulic efficiency in centrifugal pumps is requested at constant design point, it has been shown how to link the pressure differences between inlet and outlet to hydrodynamic forces and blade volume to enter incomplete sensitivities validity domain. To improve the incomplete gradients reduced order physical models have been used in order to keep the complexity of the calculation low. It has been shown that minimization of complex configurations in industrial environment is possible at very low cost using quasi-Newton algorithms and incomplete sensitivities. In particular, the optimization method has been applied to a centrifugal pump using Fine/Turbo flow solver for the state equations. Incomplete gradients have been compared to finite differences and it has been shown that the difference is mainly in the amplitude and not in the directions of the gradients. The main advantages of this approach are therefore: cheap optimization (basically design at the cost of one simulation to steady state), easy implementation in black-box environments, no extra coding and useful in multi-criteria optimization where individual sensitivities are necessary for robustness quantification.

References