Advanced Vibrations

Distributed-Parameter Systems: Approximate Methods

Lecture 20

By: H. Ahmadian

ahmadian@iust.ac.ir

School of Mechanical Engineering
Iran University of Science and Technology

Distributed-Parameter Systems: Approximate Methods

- Rayleigh's Principle
- The Rayleigh-Ritz Method
- An Enhanced Rayleigh-Ritz Method
- The Assumed-Modes Method: System Response
- The Galerkin Method
- The Collocation Method

The Assumed-Modes Method: System Response

$$y(x,t) = \sum_{i=1}^{n} \phi_i(x) q_i(t)$$

known trial functions

$$T(t) = \frac{1}{2}\int_0^L m(x)\dot{y}^2(x,t)dx = \frac{1}{2}\int_0^L m(x)\sum_{i=1}^{n}\phi_i(x)\dot{q}_i(t)\sum_{j=1}^{n}\phi_j(x)\dot{q}_j(t)dx$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\dot{q}_i(t)\dot{q}_j(t)\int_0^L m(x)\phi_i(x)\phi_j(x)dx = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}\dot{q}_i(t)\dot{q}_j(t)$$

$$m_{ij} = m_{ji} = \int_0^L m(x)\phi_i(x)\phi_j(x)dx,\ i,j = 1,2,\ldots,n$$

The Assumed-Modes Method: System Response

$$V(t) = \frac{1}{2}\int_0^L EI(x)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]^2 dx + \frac{1}{2}ky^2(L,t)$$

$$V(t) = \frac{1}{2}\int_0^L EI(x)\sum_{i=1}^{n}\frac{d^2\phi_i(x)}{dx^2}q_i(t)\sum_{j=1}^{n}\frac{d^2\phi_j(x)}{dx^2}q_j(t)dx$$

$$+\frac{1}{2}k\sum_{i=1}^{n}\phi_i(L)q_i(t)\sum_{j=1}^{n}\phi_j(L)q_j(t)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}q_i(t)q_j(t)\left[\int_0^L EI(x)\frac{d^2\phi_i(x)}{dx^2}\frac{d^2\phi_j(x)}{dx^2}dx + k\phi_i(L)\phi_j(L)\right]$$

$$=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}q_i(t)q_j(t)$$

The Assumed-Modes Method: System Response

$$\overline{\delta W}_{nc}(t) = \int_0^L f(x,t)\delta y(x,t)dx = \int_0^L f(x,t)\sum_{i=1}^{n}\phi_i(x)\delta q_i(t)dx = \sum_{i=1}^{n} Q_{inc}(t)\delta q_i(t)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k, \; k = 1, 2, \ldots, n$$

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_{j=1}^{n} m_{kj}\dot{q}_j, \; \frac{\partial V}{\partial q_k} = \sum_{j=1}^{n} k_{kj}q_j,$$

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j(t) + \sum_{j=1}^{n} k_{ij}q_j(t) = Q_i(t),$$

The Assumed-Modes Method: System Response

Example: Use the assumed-modes method in conjunction with a three-term series

$$\phi_i(x) = \sin \beta_i x$$

$$\beta_1 L = 2.215707, \ \beta_2 L = 5.032218, \ \beta_3 L = 8.057941$$

to obtain the response of the tapered rod of previous Example to the uniformly distributed force

$$f(x, t) = f_0 u(t)$$

The Assumed-Modes Method: System Response

$$M^{(3)}\ddot{\mathbf{q}}(t) + K^{(3)}\mathbf{q}(t) = \mathbf{Q}(t)$$

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} 2.783074 & 0.836697 & -0.247107 \\ 0.836697 & 13.223631 & 2.623716 \\ -0.247107 & 2.623716 & 33.078693 \end{bmatrix}$$

$$M^{(3)} = mL\begin{bmatrix} 0.563196 & 0.085462 & -0.020523 \\ 0.085462 & 0.513392 & 0.070501 \\ -0.020523 & 0.070501 & 0.505321 \end{bmatrix}$$

$$Q_i(t) = \int_0^L f(x,t)\phi_i(x)dx = f_0 u(t)\int_0^L \sin\beta_i x\,dx =$$

$$\frac{f_0 u(t)(1-\cos\beta_i L)}{\beta_i}, \quad i = 1,2,3$$

The Assumed-Modes Method: System Response

$$\mathbf{q}(t) = U\boldsymbol{\eta}(t)$$

$$U = [\mathbf{a}_1^{(3)}\ \mathbf{a}_2^{(3)}\ \mathbf{a}_3^{(3)}] = (mL)^{-1/2}\begin{bmatrix} 1.340184 & -0.167149 & 0.067503 \\ -0.054456 & 1.419516 & -0.155385 \\ 0.010464 & -0.053821 & 1.422089 \end{bmatrix}$$

$$\Lambda = \mathrm{diag}[(\omega_1^{(3)})^2\ (\omega_2^{(3)})^2\ (\omega_3^{(2)})^2] = \mathrm{diag}[4.909451\ \ 26.017151\ \ 66.003666]\frac{EA}{mL^2}$$

$$\ddot{\boldsymbol{\eta}}(t) + \Lambda\boldsymbol{\eta}(t) = \mathbf{N}(t)$$

$$\mathbf{N}(t) = U^T\mathbf{Q}(t) = \frac{f_0 L^{1/2}\mathcal{u}(t)}{m^{1/2}}\begin{bmatrix} 0.955753 \\ 0.130856 \\ 0.212459 \end{bmatrix}$$

The Assumed-Modes Method: System Response

$$\eta_1(t) = \frac{1}{\omega_1}\int_0^t N_1(t-\tau)\sin\omega_1\tau\,d\tau = \frac{0.955753\,f_0 L^{1/2}}{m^{1/2}\omega_1}\int_0^t u(t-\tau)\sin\omega_1\tau\,d\tau$$

$$= \frac{0.955753\,f_0 L^{1/2}}{m^{1/2}\omega_1^2}(1-\cos\omega_1 t)$$

$$\eta_2(t) = \frac{1}{\omega_2}\int_0^t N_2(t-\tau)\sin\omega_2\tau\,d\tau = \frac{0.130856\,f_0 L^{1/2}}{m^{1/2}\omega_2}\int_0^t u(t-\tau)\sin\omega_2\tau\,d\tau$$

$$= \frac{0.130856\,f_0 L^{1/2}}{m^{1/2}\omega_2^2}(1-\cos\omega_2 t)$$

$$\eta_3(t) = \frac{1}{\omega_3}\int_0^t N_3(t-\tau)\sin\omega_3\tau\,d\tau = \frac{0.212459\,f_0 L^{1/2}}{m^{1/2}\omega_3}\int_0^t u(t-\tau)\sin\omega_3\tau\,d\tau$$

$$= \frac{0.212459\,f_0 L^{1/2}}{m^{1/2}\omega_3^2}(1-\cos\omega_3 t)$$

$$u(x,t) = \sum_{i=1}^{3} \phi_i(x) q_i(t) = \sum_{i=1}^{3} \sin \beta_i x \sum_{j=1}^{3} U_{ij} \eta_j(t)$$

$$\begin{aligned}
= \frac{f_0 L^2}{EA} \Bigg\{ & \sin 2.215707x \Big[0.260902 \left(1 - \cos 2.215728 \sqrt{EA/mL^2} t\right) \\
& - 0.000841 \left(1 - \cos 5.100701 \sqrt{EA/mL^2} t\right) \\
& + 0.000217 \left(1 - \cos 8.124264 \sqrt{EA/mL^2} t\right) \Big] \\
& + \sin 5.032218x \Big[-0.010601 \left(1 - \cos 2.215728 \sqrt{EA/mL^2} t\right) \\
& + 0.007140 \left(1 - \cos 5.100701 \sqrt{EA/mL^2} t\right) \\
& - 0.000500 \left(1 - \cos 8.124264 \sqrt{EA/mL^2} t\right) \Big] \\
& + \sin 8.057941x \Big[0.002037 \left(1 - \cos 2.215728 \sqrt{EA/mL^2} t\right) \\
& - 0.000271 \left(1 - \cos 5.100701 \sqrt{EA/mL^2} t\right) \\
& + 0.004578 \left(1 - \cos 8.124264 \sqrt{EA/mL^2} t\right) \Big] \Bigg\}
\end{aligned}$$

THE GALERKIN METHOD

The approximate solution is assumed in the form

$$Y^{(n)}(x) = \sum_{j=1}^{n} a_j \phi_j(x)$$

known independent comparison functions from a complete set

residual

$$\int_0^L \phi_i(x) \mathcal{R}(Y^{(n)}(x), x) dx = 0, \; i = 1, 2, \ldots, n$$

Galerkin's method is more general in scope and can be used for both conservative and non-conservative systems.

THE GALERKIN METHOD

- The residual is orthogonal to every trial function.
- As *n* increases without bounds, the residual can remain orthogonal to an infinite set of independent functions only if it tends itself to zero, or

$$\lim_{n\to\infty} \mathcal{R}(Y^{(n)}(x), x) = 0, \ 0 < x < L$$

$$\lim_{n\to\infty} Y^{(n)}(x) = Y(x)$$

- Demonstrates the convergence of Galerkin's method.

THE GALERKIN METHOD

Consider a viscously damped beam in transverse vibration.

$$m(x)\frac{\partial^2 y(x,t)}{\partial t^2}+c(x)\frac{\partial y(x,t)}{\partial t}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=0,\ 0<x<L$$

$$y(x,t)=e^{\lambda t}Y(x)$$

$$\lambda^2 m(x)Y(x)+\lambda c(x)Y(x)+\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=0,$$

$$\mathcal{R}(Y^{(n)}(x),x)=(\lambda^{(n)})^2 m(x)\sum_{j=1}^{n}a_j\phi_j(x)+\lambda^{(n)}c(x)\sum_{j=1}^{n}a_j\phi_j(x)$$

$$+\sum_{j=1}^{n}a_j\frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_j(x)}{dx^2}\right],\ 0<x<L$$

THE GALERKIN METHOD

$$(\lambda^{(n)})^2 \sum_{j=1}^{n} a_j \int_0^L m(x)\phi_i(x)\phi_j(x)dx + \lambda^{(n)} \sum_{j=1}^{n} a_j \int_0^L c(x)\phi_i(x)\phi_j(x)dx$$

$$+ \sum_{j=1}^{n} a_j \int_0^L \phi_i(x) \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_j(x)}{dx^2}\right]dx = 0, \; i = 1, 2, \ldots, n$$

$$c_{ij} = c_{ji} = \int_0^L c(x)\phi_i(x)\phi_j(x)dx,$$

$$(\lambda^{(n)})^2 \sum_{j=1}^{n} m_{ij}a_j + \lambda^{(n)} \sum_{j=1}^{n} c_{ij}a_j + \sum_{j=1}^{n} k_{ij}a_j = 0,$$

$$(\lambda^{(n)})^2 M^{(n)}\mathbf{a}^{(n)} + \lambda^{(n)} C^{(n)}\mathbf{a}^{(n)} + K^{(n)}\mathbf{a}^{(n)} = \mathbf{0}$$

THE COLLOCATION METHOD

- The main difference between the collocation method and Galerkin's method lies in the weighting functions,
  - the collocation method represent spatial Dirac delta functions.

$$\int_0^L \delta(x - x_i)\mathcal{R}(Y^{(n)}(x), x)dx = 0,$$

$$\mathcal{R}(Y^{(n)}(x_i)) = 0,\ i = 1, 2, \ldots n$$

THE COLLOCATION METHOD:
A beam in transverse vibration

$$\mathcal{R}(Y^{(n)}(x_i)) = \sum_{j=1}^{n} a_j \left\{ \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_j(x)}{dx^2}\right] - \lambda^{(n)} m(x)\phi_j(x) \right\}\Bigg|_{x=x_i} = 0,$$

$$\sum_{j=1}^{n} k_{ij} a_j = \lambda^{(n)} \sum_{j=1}^{n} m_{ij} a_j, \quad i = 1, 2, \ldots, n$$

$$m_{ij} = m(x_i)\phi_j(x_i) \qquad k_{ij} = \frac{d^2}{dx^2}\left[EI(x)\frac{d^2\phi_j(x)}{dx^2}\right]\Bigg|_{x=x_i}$$

$$K^{(n)}\mathbf{a}^{(n)} = \lambda^{(n)} M^{(n)}\mathbf{a}^{(n)}$$

THE COLLOCATION METHOD:
The tapered rod

- Consider the tapered rod fixed at $x=0$ and spring-supported at x=L. Solve the problem by the collocation method in two different ways:
  - *1)* using the locations $x_i = iL/n$ $(i = 1,2, \ldots, n)$
  - *2)* using the locations $x_i=(2i-1)L/2n$ $(i=1,2,\ldots, n)$
- Give results for $n = 2$ and $n = 3$.
- List the three lowest natural frequencies for $n = 2,3, \ldots ,30$ and discuss the nature of the convergence for both cases.

THE COLLOCATION METHOD:
The tapered rod

$$\phi_i(x) = \sin\beta_i x, \quad i = 1, 2, \ldots, n$$

$$\beta_1 L = 2.215707, \quad \beta_2 L = 5.032218, \quad \beta_3 L = 8.057941$$

$$k_{ij} = -\frac{d}{dx}\left[EA(x)\frac{d\phi_j(x)}{dx}\right]\Bigg|_{x=x_i} = -\frac{6EA}{5L}\frac{d}{dx}\left\{\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\beta_j L\cos\beta_j x\right\}\Bigg|_{x=x_i}$$

$$= \frac{6EA}{5L^2}\left\{\beta_j x_i\cos\beta_j x_i + \left[1-\frac{1}{2}\left(\frac{x_i}{L}\right)^2\right](\beta_j L)^2\sin\beta_j x_i\right\}, \quad i, j = 1, 2, \ldots, n$$

$$m_{ij} = \frac{6m}{5}\left[1-\frac{1}{2}\left(\frac{x_i}{L}\right)^2\right]\sin\beta_j x_i, \quad i, j = 1, 2, \ldots, n$$

THE COLLOCATION METHOD:
The tapered rod

1. *Locations at* $x_i = iL/n$

$$K = \frac{EA}{L^2}\begin{bmatrix} 5.205939 & 13.120091 \\ 0.755692 & -12.524855 \end{bmatrix} \qquad M = m\begin{bmatrix} 0.939479 & 0.614764 \\ 0.479492 & -0.569574 \end{bmatrix}$$

$$A = (mL^2/EA)M^{-1}K \qquad A = \begin{bmatrix} 4.132826 & -0.273503 \\ 2.152427 & 21.759635 \end{bmatrix}$$

$$\lambda_1 = 4.166287\frac{EA}{mL^2}, \quad \mathbf{a}_1 = \begin{bmatrix} 0.992599 \\ -0.121438 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 0.999879 \\ 0.015544 \end{bmatrix}$$

$$\lambda_2 = 21.72617\frac{EA}{mL^2}, \quad \mathbf{a}_2 = \begin{bmatrix} 0.015544 \\ -0.999879 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 0.121438 \\ 0.992599 \end{bmatrix}$$

$$U_1(x) = 0.992599\sin 2.215707x - 0.121438\sin 5.032218x$$

$$U_2(x) = 0.015544\sin 2.215707x - 0.999879\sin 5.032218x$$

THE COLLOCATION METHOD:
The tapered rod

$$\omega_r^* = \omega_r \sqrt{mL^2/EA} \text{ for } x_i = iL/n$$

r	ω_1^*	ω_2^*	ω_3^*
2	2.041149	4.661134	–
3	2.148223	4.950458	7.764421
4	2.180078	5.026274	7.974473
5	2.193677	5.055835	8.039231
6	2.200720	5.070436	8.067294
7	2.204835	5.078739	8.082175
8	2.207446	5.083920	8.091089
9	2.209205	5.087374	8.096877
10	2.210447	5.089793	8.100861
⋮	⋮	⋮	⋮
30	2.214987	5.098508	8.114744

THE COLLOCATION METHOD:
The tapered rod

$$\omega_r^* = \omega_r \sqrt{EA/mL^2} \text{ for } x_i = (2i-1)L/2n$$

r	ω_1^*	ω_2^*	ω_3^*
2	2.245588	5.229317	—
3	2.231022	5.138904	8.255577
4	2.225251	5.121013	8.163432
5	2.222239	5.113512	8.142789
6	2.220445	5.109469	8.133890
7	2.219286	5.106992	8.129019
8	2.218494	5.105352	8.125996
9	2.217927	5.104204	8.123967
10	2.217509	5.103367	8.122529
⋮	⋮	⋮	⋮
30	2.215772	5.099994	8.117046

THE COLLOCATION METHOD:
The tapered rod

- For $x_i = iL/n$ the natural frequencies increase as n increases:
  - The specified locations tend to make the rod longer than it actually is.
  - Because an increased length, while everything else remains the same, tends to reduce the stiffness,
  - The approximate natural frequencies are lower than the actual natural frequencies.

THE COLLOCATION METHOD:
The tapered rod

- On the other hand, the locations x_i=(2i-1)L/2n tend to make the rod shorter than it actually is,
  - So that the stiffness of the model is larger than the stiffness of the actual system.
  - As a result, the approximate natural frequencies are larger than the actual natural frequencies.

This points to the arbitrariness and lack of predictability inherent in the collocation method, with the nature of the results depending on the choice of locations.

Distributed-Parameter Systems: Approximate Methods

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Advanced Vibrations

THE FINITE ELEMENT METHOD

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By: H. Ahmadian
ahmadian@iust.ac.ir

School of Mechanical Engineering
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INTRODUCTION TO THE FINITE ELEMENT METHOD

- Finite element method is the most important development in the static and dynamic analysis of structures in the second half of the twentieth century.
- Although the finite element method was developed independently, it was soon recognized as the most important variant of the Rayleigh-Ritz method.

INTRODUCTION TO THE FINITE ELEMENT METHOD

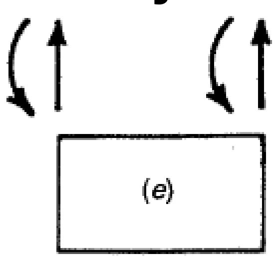
- As with the classical Rayleigh-Ritz method, the finite element method also envisions approximate solutions to problems of vibrating distributed systems in the form of linear combinations of known trial functions.
- Moreover, the expressions for the stiffness and mass matrices defining the eigenvalue problem are the same as for the classical Rayleigh-Ritz method.

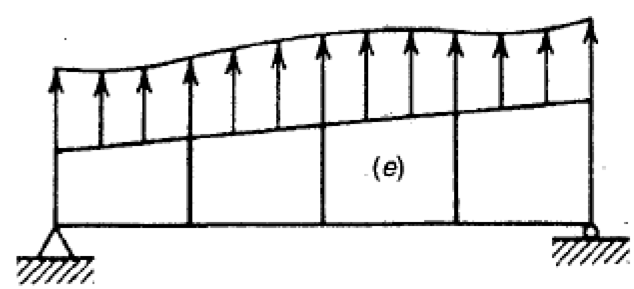
INTRODUCTION TO THE FINITE ELEMENT METHOD

- The basic difference between the two approaches lies in the nature of the trial functions.
  - in the classical Rayleigh-Ritz method the trial functions are global functions,
  - in the finite element method they are local functions extending over small sub-domains of the system, namely, over finite elements.

INTRODUCTION TO THE FINITE ELEMENT METHOD

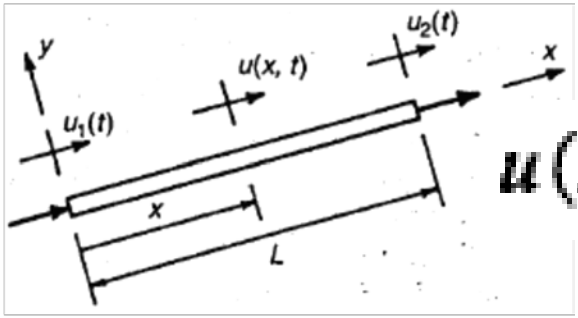
- In finite element modeling deflection shapes are limited to a portion (finite element) of the structure, with the elements being assembled to for the structural system.
- The elements are joined together at nodes, or joints, and displacement compatibility is enforced at these joints.





ELEMENT STIFFNESS AND MASS MATRICES AND FORCE VECTOR

Uniform bar element undergoing axial deformation:



$$u(x,t) = \psi_1(x)u_1(t) + \psi_2(x)u_2(t)$$

The *shape functions must satisfy the BCs:*

$$\psi_1(0) = 1, \qquad \psi_1(L) = 0$$

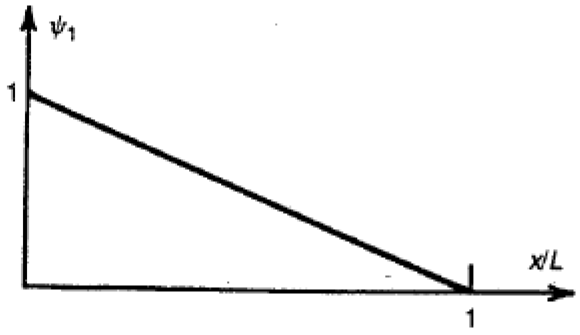
$$\psi_2(0) = 0, \qquad \psi_2(L) = 1$$

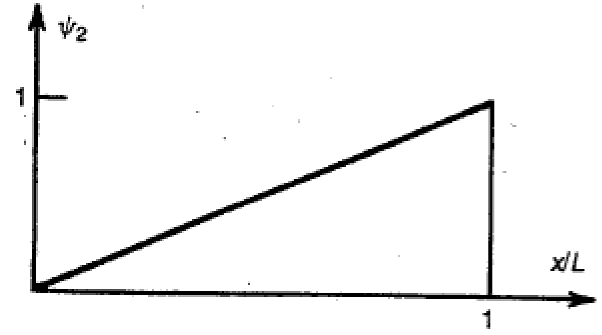
ELEMENT STIFFNESS AND MASS MATRICES AND FORCE VECTOR

- Considering axial deformation of the uniform element under static loads:

$$(AEu')' = 0 \longrightarrow u(x) = c_1 + c_2\frac{x}{L}$$

$$\boxed{\psi_1(x) = 1 - \frac{x}{L}, \qquad \psi_2(x) = \frac{x}{L}}$$





ELEMENT STIFFNESS AND MASS MATRICES AND FORCE VECTOR

$$k_{ij} = \int_0^L EA\psi_i'\psi_j'\,dx, \qquad m_{ij} = \int_0^L \rho A\psi_i\psi_j\,dx, \qquad p_i(t) = \int_0^L p_x(x,t)\psi_i\,dx$$

$$\mathbf{k} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \qquad \mathbf{m} = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Transverse Motion: Bernoulli-Euler Beam Theory

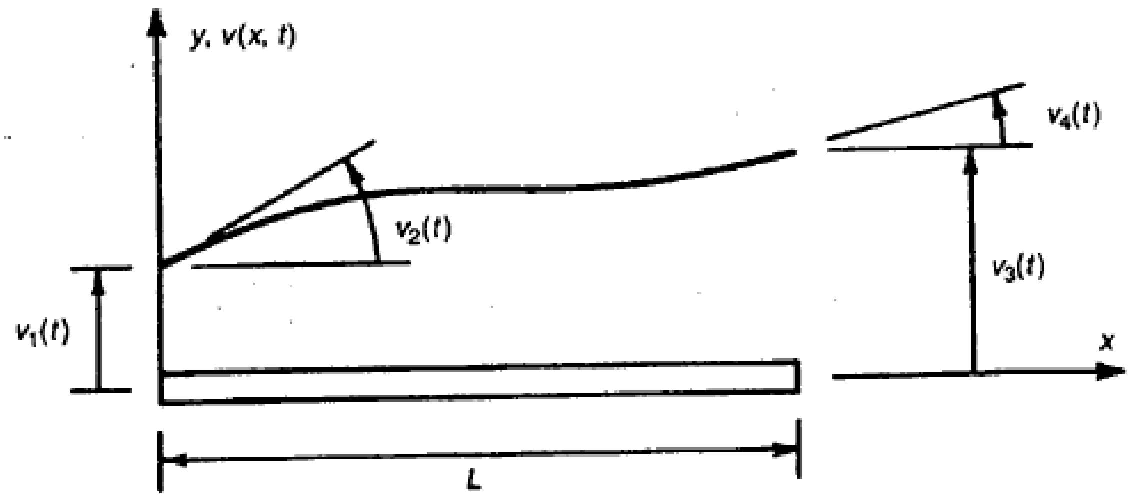
$$v(x,t) = \sum_{i=1}^{4} \psi_i(x) v_i(t)$$

$$\psi_1(0) = 1, \qquad \psi_1'(0) = \psi_1(L) = \psi_1'(L) = 0$$

$$\psi_2'(0) = 1, \qquad \psi_2(0) = \psi_2(L) = \psi_2'(L) = 0$$

$$\psi_3(L) = 1, \qquad \psi_3(0) = \psi_3'(0) = \psi_3'(L) = 0$$

$$\psi_4'(L) = 1, \qquad \psi_4(0) = \psi_4'(0) = \psi_4(L) = 0$$



Transverse Motion: Bernoulli-Euler Beam Theory

For a beam loaded only at its ends, the equilibrium equation is $(EI\,v'')'' = 0$

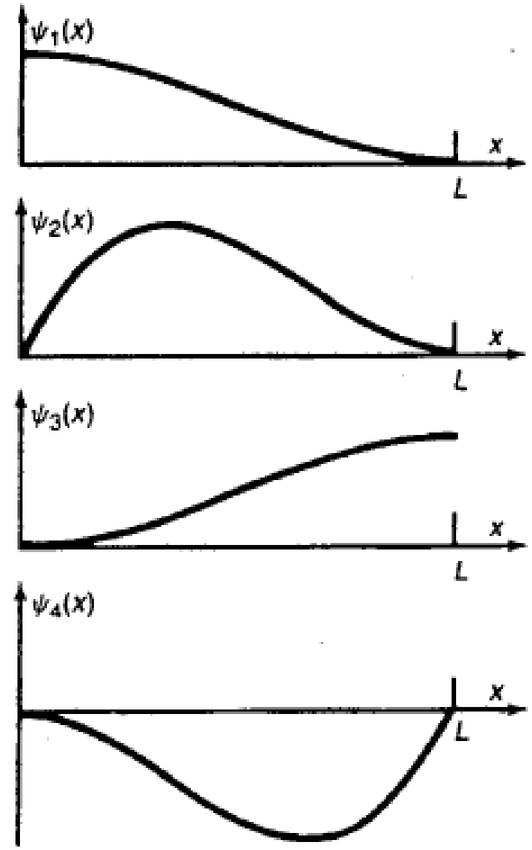
$$v(x) = c_1 + c_2\frac{x}{L} + c_3\left(\frac{x}{L}\right)^2 + c_4\left(\frac{x}{L}\right)^3$$

$$\psi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$\psi_2(x) = x - 2L\left(\frac{x}{L}\right)^2 + L\left(\frac{x}{L}\right)^3$$

$$\psi_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$\psi_4(x) = -L\left(\frac{x}{L}\right)^2 + L\left(\frac{x}{L}\right)^3$$



Transverse Motion: Bernoulli-Euler Beam Theory

$$k_{ij} = \int_0^L EI\,\psi_i''\psi_j''\,dx$$

$$m_{ij} = \int_0^L \rho A\,\psi_i\psi_j\,dx$$

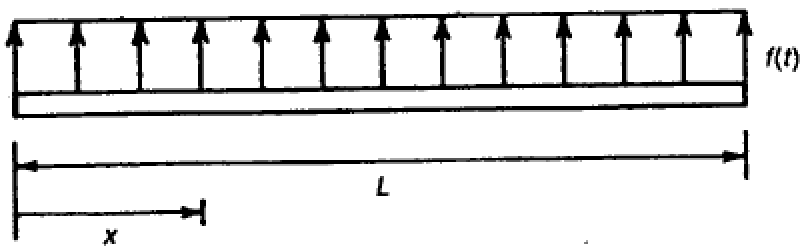
$$p_i(t) = \int_0^L p_y(x,t)\psi_i\,dx$$

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm.} & & & 4L^2 \end{bmatrix}$$

$$\mathbf{m} = \frac{\rho AL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{symm.} & & & 4L^2 \end{bmatrix}$$

Example

Determine the generalized load vector *for a beam element subjected* to a uniform transverse load.



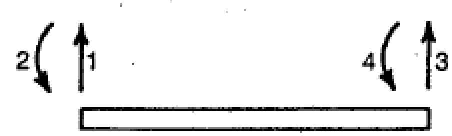
$$\psi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$

$$\psi_2(x) = x - 2L\left(\frac{x}{L}\right)^2 + L\left(\frac{x}{L}\right)^3$$

$$\psi_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$

$$\psi_4(x) = -L\left(\frac{x}{L}\right)^2 + L\left(\frac{x}{L}\right)^3$$

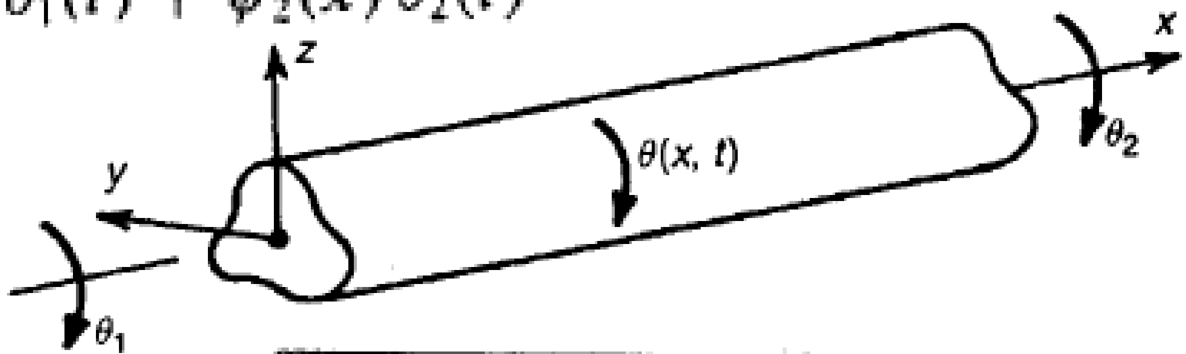
$$p_i(t) = \int_0^L p_y(x, t)\psi_i(x)\,dx$$



$$\mathbf{p}(t) = \begin{Bmatrix} \frac{1}{2} f(t) L \\ \frac{1}{12} f(t) L^2 \\ \frac{1}{2} f(t) L \\ -\frac{1}{12} f(t) L^2 \end{Bmatrix}$$

Torsion

$$\theta(x,t) = \psi_1(x)\,\theta_1(t) + \psi_2(x)\,\theta_2(t)$$



$$(GJ\theta')' = 0 \longrightarrow \boxed{\psi_1(x) = 1 - \frac{x}{L}, \qquad \psi_2(x) = \frac{x}{L}}$$

$$k_{ij} = \int_0^L GJ\,\psi_i'\psi_j'\,dx$$

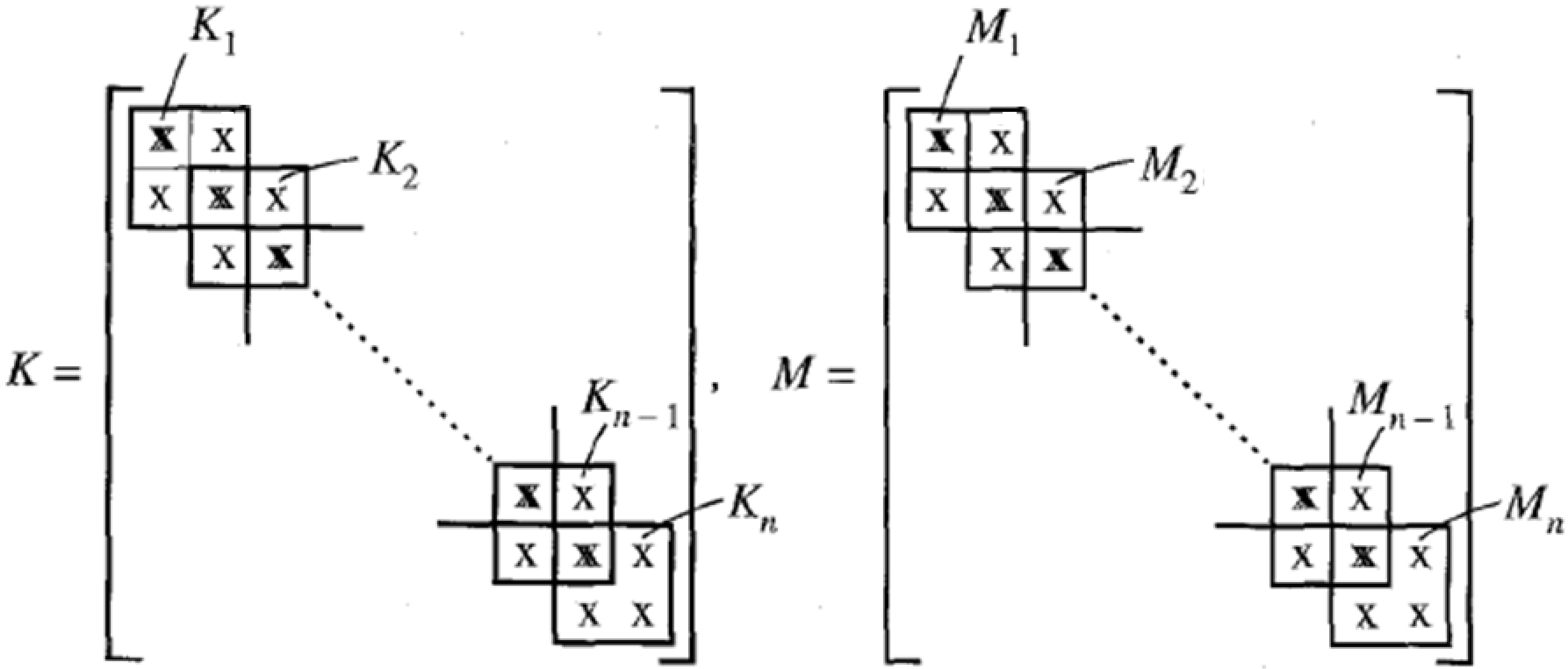
$$m_{ij} = \int_0^L \rho I_p \psi_i \psi_j\,dx$$

$$p_i(t) = \int_0^L t_\theta(x,t)\psi_i\,dx$$

$$\boxed{\mathbf{k} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{m} = \frac{\rho I_p L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$

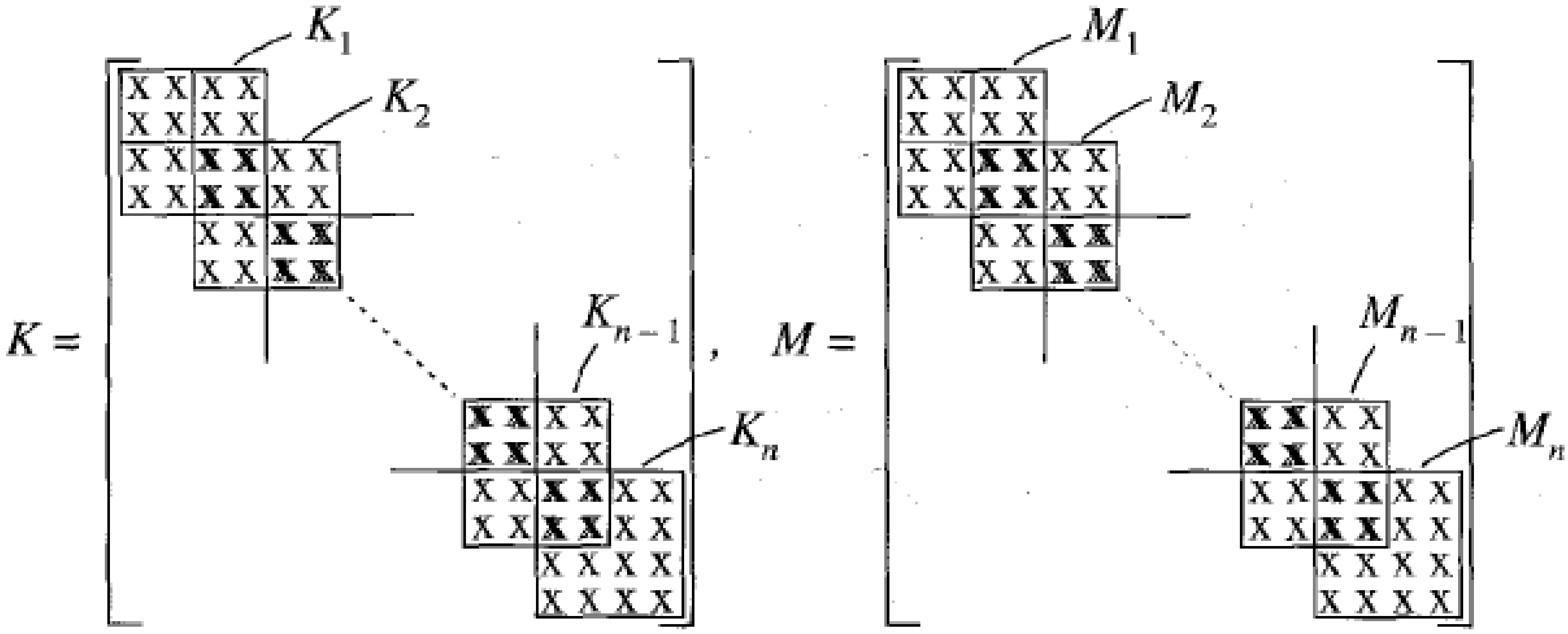
ASSEMBLY OF SYSTEM MATRICES:

- Scheme for the assembly of global matrices from element matrices for second-order systems using linear interpolation functions



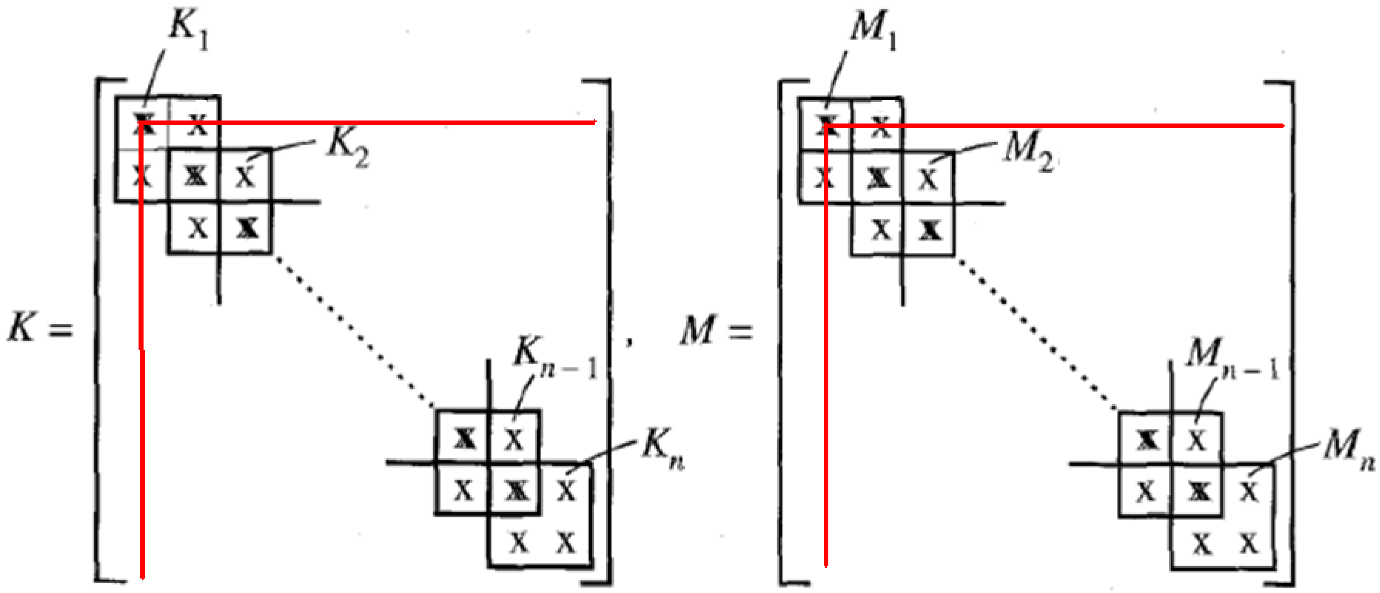
ASSEMBLY OF SYSTEM MATRICES:

- Scheme for the assembly of global matrices from element matrices for fourth-order systems



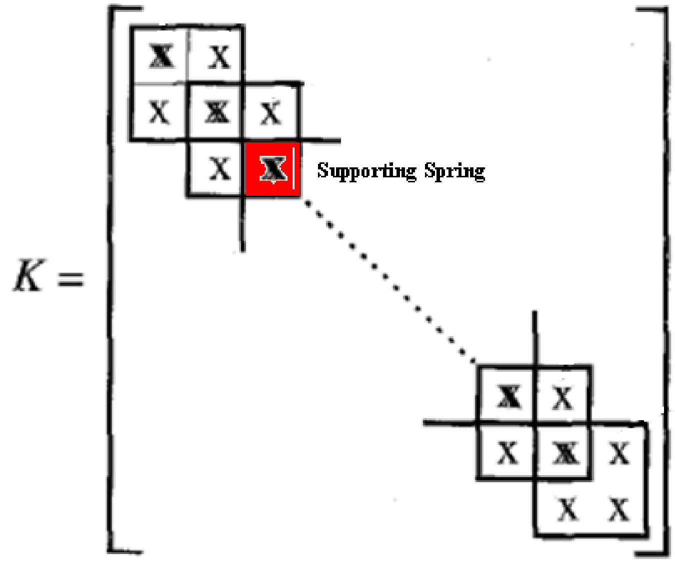
BOUNDARY CONDITIONS

- The Finite Element formulations inherently satisfy Free boundary conditions.
- Fixed BC's:

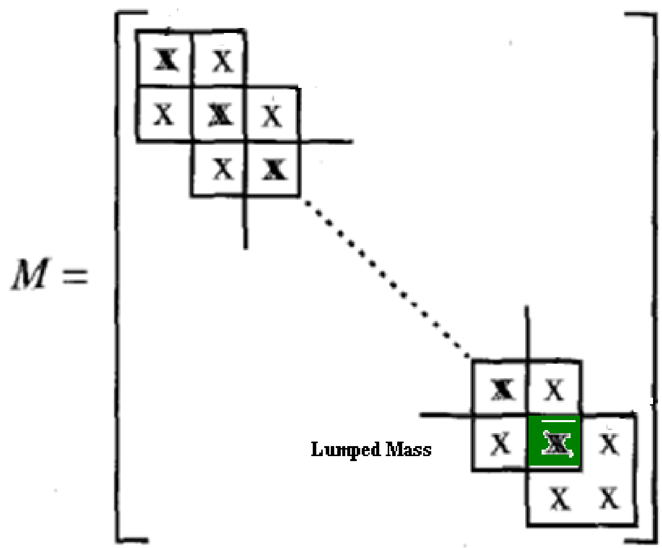


BOUNDARY CONDITIONS

- Supported Spring :



- Lumped Mass:



Example 10.1:The eigenvalue problem for the tapered rod in axial vibration

1. Use the element stiffness and mass matrices with variable cross sections given by:

$$m(x) = \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right], \quad EA(x) = \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

2. Approximate the stiffness and mass distributions over the finite elements (piece wise constant) as:

$$EA_j = \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{2j-1}{2n}\right)^2\right], \quad m_j = \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{2j-1}{2n}\right)^2\right], \quad j = 1, 2, \ldots, n$$

Example 10.1:Variable cross section rod element

$$U(x) = \phi_j^T(x)\mathbf{a}_j,\ (j-1)h < x < jh$$
$$= [\phi_{j-1}(x) \quad \phi_j(x)]\ [a_{j-1} \quad a_j]^T$$

$$\phi_1(\xi) = \xi,\ \phi_2(\xi) = 1-\xi$$

$$\xi = (jh - x)/h$$

$$\int_{(j-1)h}^{jh} EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx$$

$$= \int_1^0 EA_j(\xi)\mathbf{a}_j^T\left(-\frac{1}{h}\right)^2 \frac{d\phi(\xi)}{d\xi}\frac{d\phi^T(\xi)}{d\xi}\mathbf{a}_j(-h)d\xi$$

$$= \mathbf{a}_j^T\left(\frac{1}{h}\int_0^1 EA_j(\xi)\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T d\xi\right)\mathbf{a}_j$$

$$= \mathbf{a}_j^T\left(\frac{1}{h}\int_0^1 EA_j(\xi)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^T d\xi\right)\mathbf{a}_j$$

$$= \mathbf{a}_j^T K_j \mathbf{a}_j,\ j = 1, 2, \ldots, n$$

Example 10.1:Variable cross section rod element

$$EA(\xi) = \frac{6EA}{5}\left[1 - \frac{(j-\xi)^2}{2n^2}\right], \quad j = 1, 2, \ldots, n$$

$$K_j = \frac{6EAn}{5L}\left[1 - \frac{1-3j+3j^2}{6n^2}\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Example 10.1:Variable cross section rod element

$$\int_{(j-1)h}^{jh} m(x)U^2(x)dx = \int_1^0 m_j(\xi)\mathbf{a}_j^T\phi(\xi)\phi^T(\xi)\mathbf{a}_j(-h)d\xi$$

$$= \mathbf{a}_j^T\left(h\int_0^1 m_j(\xi)\begin{bmatrix}\xi \\ 1-\xi\end{bmatrix}\begin{bmatrix}\xi \\ 1-\xi\end{bmatrix}^T d\xi\right)\mathbf{a}_j$$

$$= \mathbf{a}_j^T\left(h\int_0^1 m_j(\xi)\begin{bmatrix}\xi^2 & \xi(1-\xi) \\ \xi(1-\xi) & (1-\xi)^2\end{bmatrix}d\xi\right)\mathbf{a}_j$$

$$= \mathbf{a}_j^T M_j\mathbf{a}_j, \quad j = 1, 2, \ldots, n$$

$$M_j = h\int_0^1 m_j(\xi)\begin{bmatrix}\xi^2 & \xi(1-\xi) \\ \xi(1-\xi) & (1-\xi)^2\end{bmatrix}d\xi, \qquad m(\xi) = \frac{6m}{5}\left[1 - \frac{(j-\xi)^2}{2n^2}\right],$$

$$M_j = \frac{mL}{5n}\begin{bmatrix}2 & 1 \\ 1 & 2\end{bmatrix} - \frac{mL}{100n^3}\begin{bmatrix}2(6-15j+10j^2) & 3-10j+10j^2 \\ 3-10j+10j^2 & 2(1-5j+10j^2)\end{bmatrix},$$

Example 10.1:Assembelled Stiffness Matrix

$$K=\frac{6EAn}{5L}\begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 1+5/6n \end{bmatrix}$$

the spring at $x = L$ has the stiffness $k = EA/L$

$$-\frac{EA}{5Ln}\begin{bmatrix} 8 & -7 & 0 & \dots & 0 & 0 \\ -7 & 26 & -19 & \dots & 0 & 0 \\ 0 & -19 & 56 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & 2(4-6n+3n^2) & -(1-3n+3n^2) \\ 0 & 0 & 0 & \dots & -(1-3n+3n^2) & 1-3n+3n^2 \end{bmatrix}$$

Example 10.1:Assembelled Mass Matrix

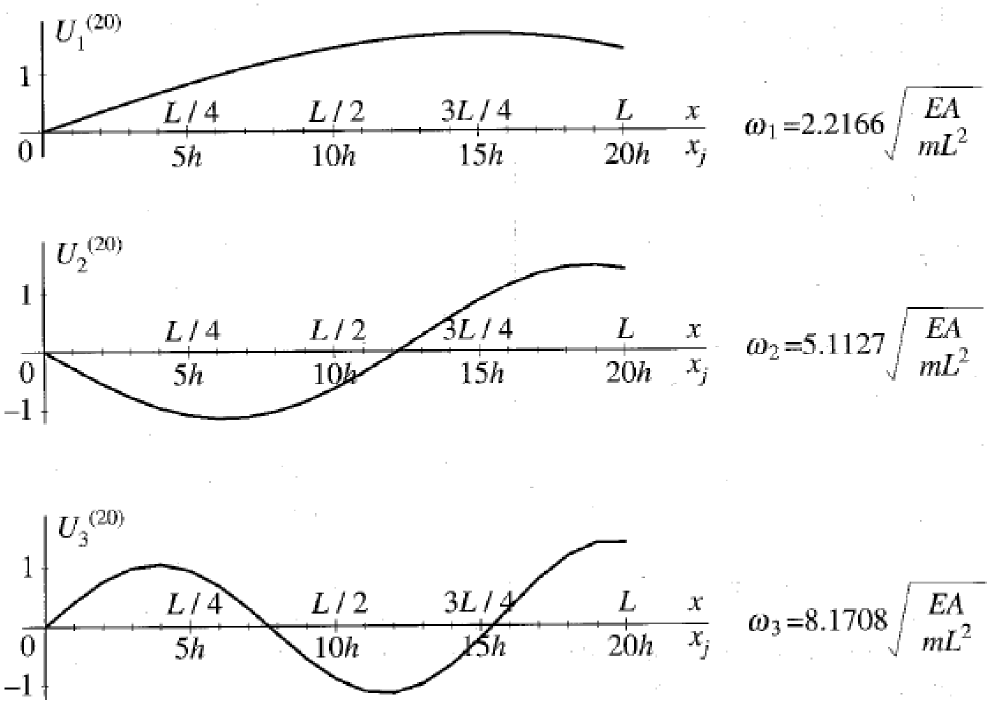
$$M = \frac{mL}{5n}\begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 4 & 1 \\ 0 & 0 & 0 & \dots & 1 & 2 \end{bmatrix}$$

$$- \frac{mL}{100n^3}\begin{bmatrix} 44 & 23 & 0 & \dots & 0 & 0 \\ 23 & 164 & 63 & \dots & 0 & 0 \\ 0 & 63 & 364 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 4(11-20n+10n^2) & 3-10n+10n^2 \\ 0 & 0 & 0 & \dots & 3-10n+10n^2 & 2(1-5n+10n^2) \end{bmatrix}$$

Example 10.1:Exact Parameter Distributions

Table 10.1 Normalized Natural Frequencies for Linear Interpolation Functions—Exact Parameter Distributions

n	$\omega_1^{(n)}\sqrt{mL^2/EA}$	$\omega_2^{(n)}\sqrt{mL^2/EA}$	$\omega_3^{(n)}\sqrt{mL^2/EA}$
10	2.219979	5.152368	8.334965
11	2.219206	5.143180	8.296875
12	2.218619	5.136197	8.267934
13	2.218161	5.130764	8.245432
14	2.217798	5.126456	8.227593
⋮	⋮	⋮	⋮
20	2.216639	5.112713	8.170752
⋮	⋮	⋮	⋮
29	2.216054	5.105795	8.142184
30	2.216020	5.105384	8.140487
31	2.215988	5.105012	8.138952
⋮	⋮	⋮	⋮
73	2.215608	5.100514	8.120397
74	2.215606	5.100487	8.120288
75	2.215604	5.100462	8.120182



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Example 10.1: Convergence Study

Table 10.1 Normalized Natural Frequencies for Linear Interpolation Functions—Exact Parameter Distributions

n	$\omega_1^{(n)}\sqrt{mL^2/EA}$	$\omega_2^{(n)}\sqrt{mL^2/EA}$	$\omega_3^{(n)}\sqrt{mL^2/EA}$
10	2.219979	5.152368	8.334965
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14	2.217798	5.126456	8.227593
⋮	⋮	⋮	⋮
20	2.216639	5.112713	8.170752
⋮	⋮	⋮	⋮
29	2.216054	5.105795	8.142184
30	2.216020	5.105384	8.140487
31	2.215988	5.105012	8.138952
⋮	⋮	⋮	⋮
73	2.215608	5.100514	8.120397
74	2.215606	5.100487	8.120288
75	2.215604	5.100462	8.120182

Table 10.2 Normalized Natural Frequencies for Linear Interpolation Functions—Approximate Parameter Distributions

n	$\omega_1^{(n)}\sqrt{mL^2/EA}$	$\omega_2^{(n)}\sqrt{mL^2/EA}$	$\omega_3^{(n)}\sqrt{mL^2/EA}$
10	2.219493	5.148365	8.325987
11	2.218807	5.139915	8.289660
12	2.218285	5.133480	8.262000
13	2.217877	5.128467	8.240460
14	2.217554	5.124487	8.223362
⋮	⋮	⋮	⋮
20	2.216520	5.111767	8.168765
⋮	⋮	⋮	⋮
29	2.215998	5.105350	8.141259
30	2.215967	5.104968	8.139624
31	2.215939	5.104623	8.138144
⋮	⋮	⋮	⋮
73	2.215599	5.100444	8.120254
74	2.215597	5.100420	8.120148
75	2.215595	5.100396	8.120046

Rayleigh-Ritz method using comparison functions reach convergence as follows: $\omega_1^{(14)} = 2.215524\sqrt{EA/mL^2}$, $\omega_2^{(14)} = 5.099525\sqrt{EA/mL^2}$, $\omega_3^{(20)} = 8.116318\sqrt{EA/mL^2}$.

Advanced Vibrations

Superaccurate finite element eigenvalue computation

Lecture 22

By: H. Ahmadian

ahmadian@iust.ac.ir

School of Mechanical Engineering

Iran University of Science and Technology

Superaccurate finite element eigenvalue computation

The consistent finite element formulation:

- It is theoretically sound and also,
- provides an assured upper bound on the lowest eigenvalue.

Mass lumping producing a diagonal mass matrix

- An attractive option for the engineer confronted with large complex systems.

A Fixed-Free Rod Finite Element Code

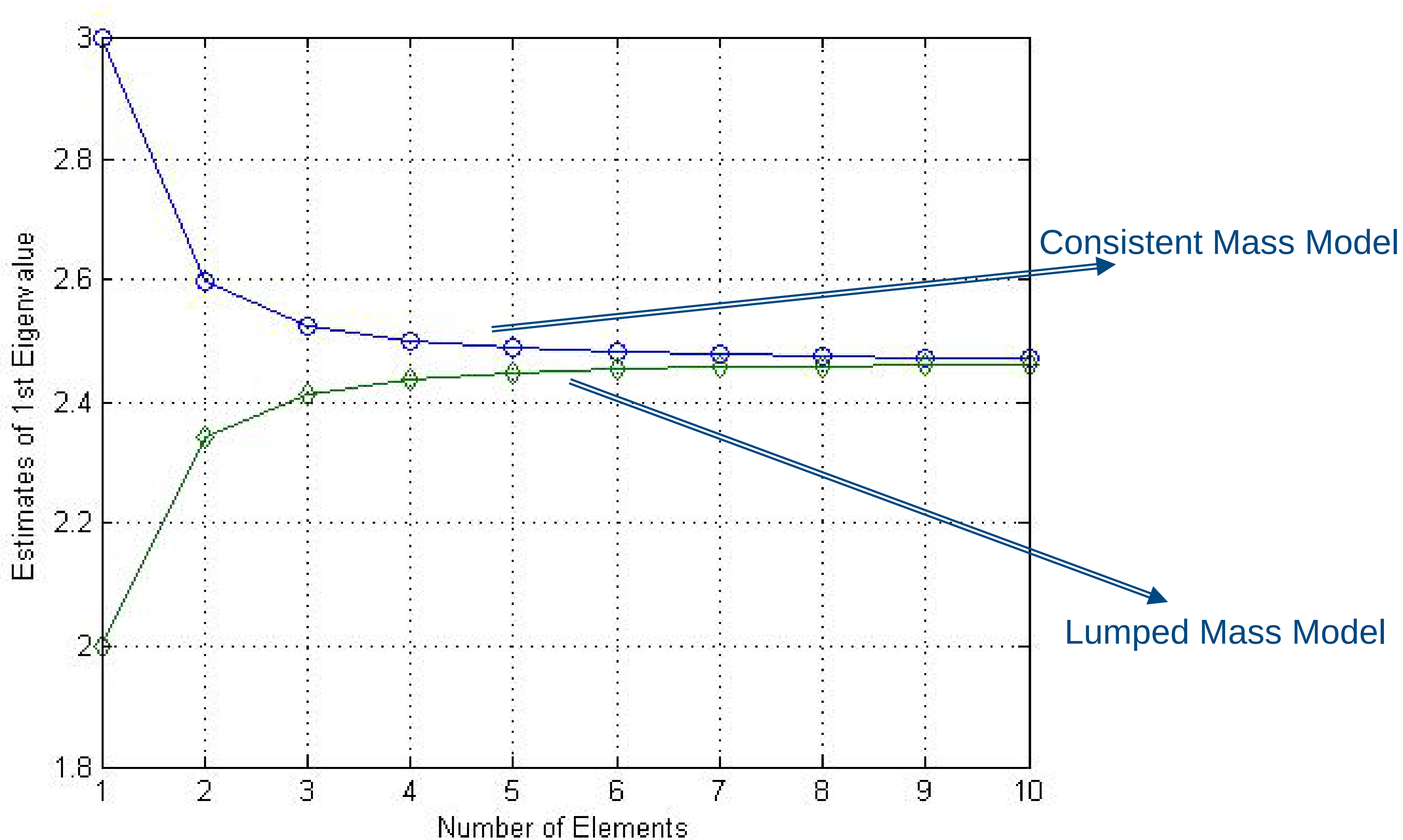
Editor - F:\COURSES\Advanced Vibrations\Rod.m

```
%
% Eigen Problem of a Fixed Free Rod
%
L=1;             % The rod length
m=1;             % Linear mass density
EA=1;            % Axial stiffness
N=10;
for Ne=1:N
    K=zeros(Ne+1); Mc=zeros(Ne+1); Ml=zeros(Ne+1);
    ke=EA*Ne/L*[1,-1;-1,1];
    me=m*L/Ne*[2,1;1,2]/6;
    ml=m*L/Ne*[1,0;0,1]/2;
    for i=1:Ne
        K(i:i+1,i:i+1)= K(i:i+1,i:i+1)+ke;
        Mc(i:i+1,i:i+1)= Mc(i:i+1,i:i+1)+me;
        Ml(i:i+1,i:i+1)= Ml(i:i+1,i:i+1)+ml;
    end
    K=K(2:Ne+1,2:Ne+1); Mc=Mc(2:Ne+1,2:Ne+1); Ml=Ml(2:Ne+1,2:Ne+1)
    Lc(Ne)=min(eig(K,Mc));
    Ll(Ne)=min(eig(K,Ml));
end
plot(1:N,Lc,1:N,Ll);
```

Consistent Mass Model

Lumped Mass Model

Convergence Study of the 1st Mode



Superaccurate finite element eigenvalue computation

$$k_e = \frac{1}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad m_e = \frac{h}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad m_e = \frac{h}{2}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Let the interval $0 \leqslant x \leqslant 1$ be divided into $n+1$ sections of size $h = 1/(n+1)$

Assembly of the linear finite elements over this mesh using the lumped mass matrix leads to:

$$u_j - 2u_{j+1} + u_{j+2} + \omega^2 h^2 u_{j+1} = 0, \quad u_0 = u_{u+1} = 0.$$

$$u_j = z^j \longrightarrow z^2 + z(-2 + \omega^2 h^2) + 1 = 0$$

Provided: $\omega^2 h^2 < 4.$ $\quad z = 1 - \frac{1}{2}\omega^2 h^2 \pm i h\omega\sqrt{1 - \frac{1}{4}\omega^2 h^2}$

Superaccurate finite element eigenvalue computation

$$z = 1 - \tfrac{1}{2}\omega^2 h^2 \pm ih\omega\sqrt{1 - \tfrac{1}{4}\omega^2 h^2}$$

$$|z| = 1 \longrightarrow z = \cos\theta \pm \mathrm{i}\sin\theta$$

$$u_j = c_1 z_1^j + c_2 z_2^j$$

$$u_j = (c_1 + c_2)\cos j\theta + \mathrm{i}(c_1 - c_2)\sin j\theta = A\cos j\theta + B\sin j\theta$$

$A = 0$ since $u_0 = 0$.

$$u_{u+1} = 0. \longrightarrow B\sin(n+1)\theta = 0$$

$$(n+1)\theta = \pi \quad \text{or} \quad \theta = \pi h.$$

Superaccurate finite element eigenvalue computation

$$z = 1 - \tfrac{1}{2}\omega^2 h^2 \pm i h\omega\sqrt{1 - \tfrac{1}{4}\omega^2 h^2}$$

$$z = \cos\theta \pm i\sin\theta \qquad \theta = \pi h.$$

$$\cos \pi h = 1 - \tfrac{1}{2}\omega^2 h^2 \qquad \omega^2 = \frac{2}{h^2}(1 - \cos \pi h).$$

Power series expansion of $\cos \pi h$ results in

$$\lambda = \pi^2\left(1 - \tfrac{1}{12}\pi^2 h^2 \pm \cdots\right)$$

an underestimation of π^2 of accuracy $O(h^2)$.

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Superaccurate finite element eigenvalue computation

Assembly of the linear finite elements with the consistent mass matrix

$$u_j + 2u_{j+1} + u_{j+2} + \tfrac{1}{6}\omega^2 h^2 (u_j + 4u_{j+1} + u_{j+2}) = 0$$

the associated characteristic equation

$$z^2 + 2\frac{-6 + 2\omega^2 h^2}{6 + \omega^2 h^2} z + 1 = 0.$$

$$z = \cos\theta \pm \mathrm{i}\sin\theta \longrightarrow \cos\pi h = \frac{6 - 2\omega^2 h^2}{6 + \omega^2 h^2}$$

$$\omega^2 = \frac{6}{h^2}\frac{1 - \cos\pi h}{2 + \cos\pi h}. \longrightarrow \omega^2 = \pi^2\left(1 + \tfrac{1}{12}\pi^2 h^2 + \tfrac{1}{360}\pi^4 h^4 + \cdots\right)$$

ω^2 is an overestimation of π^2 of the same accuracy $O(h^2)$.

Superaccurate finite element eigenvalue computation

If:

- The consistent finite element formulation leads to an overestimation of eigenvalues and
- The lumped finite element formulation leads to an underestimation of eigenvalue;

then it stands to reason that

- an intermediate formulation should exist that is accurately superior to both formulations.

Superaccurate finite element eigenvalue computation

Linear combinations of the lumped and the consistent mass matrices give various forms of nonconsistent mass matrices:

$$[M_{NC}] = \alpha [M_L] + \beta [M_c],$$

where the constraint $\alpha + \beta = 1$ is imposed for mass conservation.

Optimal element mass distribution

Write the general finite difference approximation:

$$u_j - 2u_{j+1} + u_{j+2} + \omega^2 h^2(\alpha_0 u_j + \alpha_1 u_{j+1} + \alpha_0 u_{j+2}) = 0$$

$$z^2 + 2z\frac{-1 + \frac{1}{2}\alpha_1\omega^2 h^2}{1 + \alpha_0\omega^2 h^2} + 1 = 0$$

$$\cos\pi h = \frac{2 - \alpha_1\omega^2 h^2}{2 + 2\alpha_0\omega^2 h^2} \qquad \omega^2 = \frac{1}{\alpha_0 h^2}\frac{1 - \cos\pi h}{\beta + \cos\pi h}, \qquad \beta = \frac{\alpha_1}{2\alpha_0}.$$

$$\omega^2 = \pi^2\left[\frac{1}{2\alpha_0 + \alpha_1} + \frac{10\alpha_0 - \alpha_1}{12(2\alpha_0 + \alpha_1)^2}x^2 + O(x^4)\right]$$

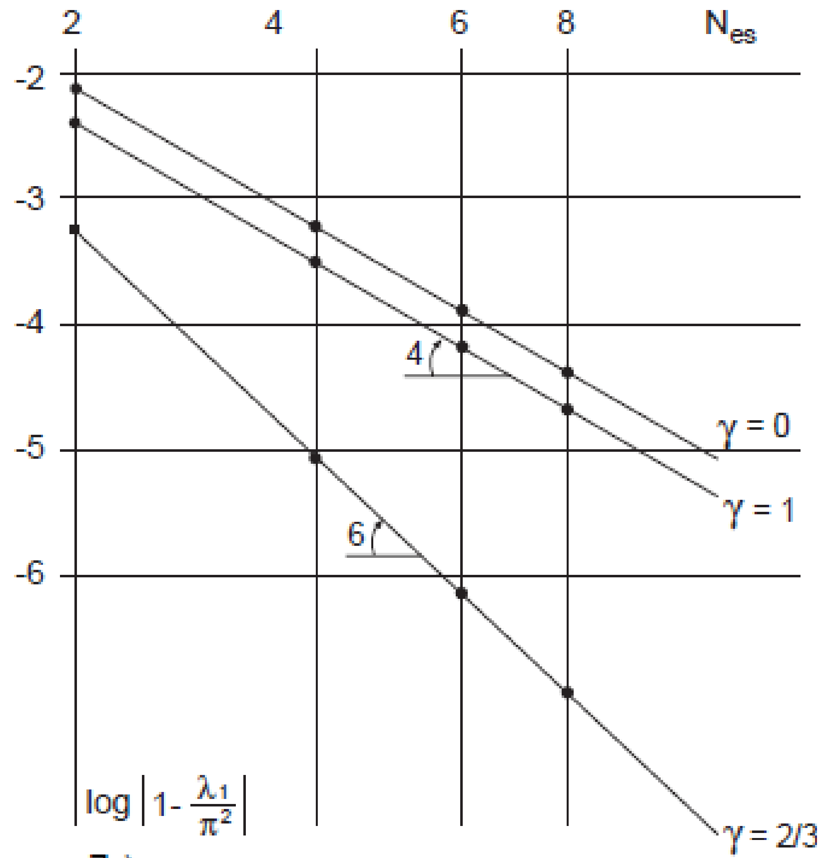
$$\alpha_0 = \frac{1}{12}, \quad \alpha_1 = \frac{10}{12},$$

Three-nodes string element

$$k_e = \frac{1}{6h}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix},$$

$$m_e = \frac{h}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix},$$

$$m_e = \frac{h}{3}\begin{bmatrix} 1 & & \\ & 4 & \\ & & 1 \end{bmatrix}$$



$$m_e(\gamma) = \frac{h}{15}\left(\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} + \gamma\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}\right)$$

More details in:

Superaccurate finite element eigenvalue computation

I. Fried*, M. Chavez

Journal of Sound and Vibration 275 (2004) 415–422

PARAMETRIC MODELS AND ERROR ANALYSIS FOR RODS

Rod parametric model:

$$\mathbf{K} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad k > 0$$

$$\mathbf{M} = \rho A \Delta x\begin{bmatrix} \frac{1}{2} - \theta & \theta \\ \theta & \frac{1}{2} - \theta \end{bmatrix}, \quad \theta < 1/4$$

$$\mathbf{K\Phi}_\mathrm{R} = 0$$

$$\mathbf{\Phi}_\mathrm{R}^\mathrm{T}\mathbf{M\Phi}_\mathrm{R} = \mathrm{diag}(m, m, m, I_{xx}, I_{yy}, I_{zz})$$

the symmetry considerations

PARAMETRIC MODELS AND ERROR ANALYSIS FOR RODS

The equation of the ith node in the assembled finite element model

$$k(-u_{i-1} + 2u_i - u_{i+1}) + \rho A \Delta x(\theta \ddot{u}_{i-1} + (1-2\theta)\ddot{u}_i + \theta \ddot{u}_{i+1}) = 0, \quad i = 2, \dots, n$$

$$\left(k\Delta x \frac{\partial^2 u_i}{\partial x^2} - \rho A \ddot{u}_i\right) + \sum_{m=1}^{\infty} \frac{2\Delta x^{2m}}{(2m)!}\left(\frac{k\Delta x}{(2m+1)(2m+2)} \frac{\partial^{2(m+1)} u_i}{\partial x^{2(m+1)}} - \rho A \theta \frac{\partial^{2m} \ddot{u}_i}{\partial x^{2m}}\right) = 0$$

$$E \frac{\partial^2 u}{\partial x^2} - \rho \ddot{u} = 0$$

$$0^{th} \Rightarrow k = EA/dx$$

$$2^{nd} \Rightarrow No \quad new \quad req.$$

$$4^{th} \Rightarrow \theta = 1/12$$

PARAMETRIC MODELS AND ERROR ANALYSIS FOR BEAMS

$$\mathbf{K} = k \begin{bmatrix} 1 & \frac{1}{2} & -1 & \frac{1}{2} \\ & \alpha & -\frac{1}{2} & \frac{1}{2} - \alpha \\ & & 1 & -\frac{1}{2} \\ \text{Sym.} & & & \alpha \end{bmatrix}$$

$$\mathbf{M} = \rho A \Delta x \begin{bmatrix} m_{1,1} & m_{1,2} & \frac{1}{2} - m_{1,1} & m_{1,4} \\ & m_{2,2} & -m_{1,4} & m_{2,4} \\ & & m_{1,1} & -m_{1,2} \\ \text{Sym.} & & & m_{2,2} \end{bmatrix}$$

$$m_{2,4} = \tfrac{1}{6} - m_{1,1}/2 + m_{1,2} + m_{1,4} - m_{2,2}$$

Timoshenki beam element,

$$k = 12 \frac{EI}{(1+g)\Delta x^3}$$

$$\alpha = (4+g)/12$$

$$m_{1,1} = (\tfrac{13}{35} + \tfrac{7}{10}g + \tfrac{1}{3}g^2)/(1+g)^2$$

$$m_{1,2} = (\tfrac{11}{210} + \tfrac{11}{120}g + \tfrac{1}{24}g^2)/(1+g)^2$$

$$m_{1,4} = -(\tfrac{13}{420} + \tfrac{3}{40}g + \tfrac{1}{24}g^2)/(1+g)^2$$

$$m_{2,2} = (\tfrac{1}{105} + \tfrac{1}{60}g + \tfrac{1}{120}g^2)/(1+g)^2$$

Euler–Bernoulli beam

$$g = 0.$$

PARAMETRIC MODELS AND ERROR ANALYSIS FOR BEAMS

New Concepts for Finite-Element Mass Matrix Formulations

AIAA JOURNAL VOL. 27, NO. 9, SEPTEMBER 1989

C. Stavrinidis*
European Space Agency, European Space Research and Technology Centre
Noordwijk, The Netherlands
and
J. Clinckemaillie† and J. Dubois‡
Engineering System International, Rungis Cédèx, France

$$m_{1,1} = \frac{163}{420}, \quad m_{1,2} = \frac{51}{840}, \quad m_{1,4} = -\frac{19}{840}, \quad m_{2,2} = \frac{15}{840}$$

The resultant mass matrix leads to an accuracy of fourth order in vibration analysis which cannot be obtained by a linear combination of the consistent and lumped models.

PARAMETRIC MODELS AND ERROR ANALYSIS FOR PLATES

MINIMIZATION OF THE DISCRETIZATION ERROR IN MASS AND STIFFNESS FORMULATIONS BY AN INVERSE METHOD

H. AHMADIAN,[1] M. I. FRISWELL[2] AND J. E. MOTTERSHEAD[1]*

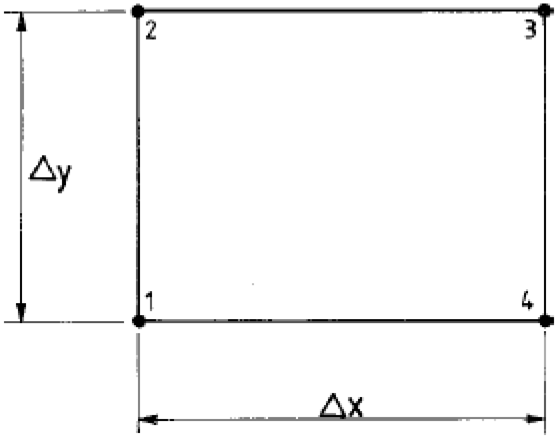
INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING, VOL. 41, 371–387 (1998)

PARAMETRIC MODELS AND ERROR ANALYSIS FOR PLATES

$$\mathbf{d} = [\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3, \mathbf{d}_4]^{\mathrm{T}}$$

$$\mathbf{d}_i = [w_i, \Delta y \alpha_i, \Delta x \beta_i]^{\mathrm{T}}, \quad i = 1, \ldots, 4$$

$$\mathbf{K} = k \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} \\ & \mathbf{K}_{22} & \mathbf{K}_{23} & \mathbf{K}_{24} \\ & & \mathbf{K}_{33} & \mathbf{K}_{34} \\ \text{Sym.} & & & \mathbf{K}_{44} \end{bmatrix}$$



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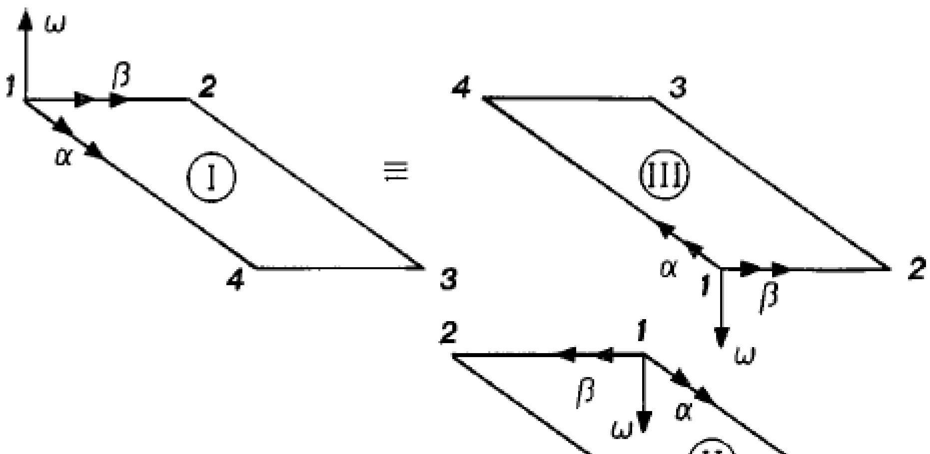

$$\underline{K}^{(I)} = \underline{K}^{(II)} = \underline{K}^{(III)}$$

$$\mathbf{T}_{xx} = \begin{bmatrix} 0 & 0 & 0 & \mathbf{R} \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ \mathbf{R} & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{T}_{yy} = \begin{bmatrix} 0 & \mathbf{S} & 0 & 0 \\ \mathbf{S} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{S} \\ 0 & 0 & \mathbf{S} & 0 \end{bmatrix}$$

$$\mathbf{R} = \mathrm{diag}(-1, -1, 1)$$

$$\mathbf{S} = \mathrm{diag}(-1, 1, -1)$$

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$$\mathbf{K} = k \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} \\ & \mathbf{S}\mathbf{K}_{11}\mathbf{S} & \mathbf{S}\mathbf{K}_{14}\mathbf{S} & \mathbf{S}\mathbf{K}_{13}\mathbf{S} \\ & & \mathbf{S}\mathbf{R}\mathbf{K}_{11}\mathbf{R}\mathbf{S} & \mathbf{S}\mathbf{R}\mathbf{K}_{12}\mathbf{R}\mathbf{S} \\ \text{Sym.} & & & \mathbf{R}\mathbf{K}_{11}\mathbf{R} \end{bmatrix}$$

$$\mathbf{K}_{11} = \mathbf{K}_{11}^{T}$$

$$\mathbf{K}_{12} = \mathbf{S}\mathbf{K}_{12}^{T}\mathbf{S}$$

$$\mathbf{K}_{13} = \mathbf{S}\mathbf{R}\mathbf{K}_{13}^{T}\mathbf{R}\mathbf{S}$$

$$\mathbf{K}_{14} = \mathbf{R}\mathbf{K}_{14}^{T}\mathbf{R}$$

$$\mathbf{T}_{zz} = \begin{bmatrix} 0 & \mathbf{Q} & 0 & 0 \\ 0 & 0 & \mathbf{Q} & 0 \\ 0 & 0 & 0 & \mathbf{Q} \\ \mathbf{Q} & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{K}_{11}(1/p) = \mathbf{Q}^{T}\mathbf{R}\mathbf{K}_{11}(p)\mathbf{R}\mathbf{Q}$$

$$\mathbf{K}_{13}(1/p) = \mathbf{Q}^{T}\mathbf{R}\mathbf{K}_{13}(p)\mathbf{R}\mathbf{Q}$$

$$\mathbf{K}_{14}(1/p) = \mathbf{Q}^{T}\mathbf{R}\mathbf{K}_{12}(p)\mathbf{R}\mathbf{Q}$$

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The rigid-body modes must occupy the null space of **K**,

$$\Phi_R = \begin{bmatrix} \mathbf{SAS} \\ \mathbf{A} \\ \mathbf{RAR} \\ \mathbf{SRARS} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{K}_{11}\mathbf{SAS} + \mathbf{K}_{12}\mathbf{A} + \mathbf{K}_{13}\mathbf{RAR} + \mathbf{K}_{14}\mathbf{SRARS} = 0$$

The stiffness matrix of the plate element with nine independent parameters

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$$\begin{aligned}
&\mathbf{K}_{31}\mathbf{d}_{i-1,j-1} + (\mathbf{K}_{32} + \mathbf{K}_{41})\mathbf{d}_{i-1,j} + \mathbf{K}_{42}\mathbf{d}_{i-1,j+1} \\
&\quad + (\mathbf{K}_{34} + \mathbf{K}_{21})\mathbf{d}_{i,j-1} + (\mathbf{K}_{11} + \mathbf{K}_{22} + \mathbf{K}_{33} + \mathbf{K}_{44})\mathbf{d}_{i,j} \\
&\quad + (\mathbf{K}_{43} + \mathbf{K}_{12})\mathbf{d}_{i,j-1} + \mathbf{K}_{24}\mathbf{d}_{i+1,j-1} + (\mathbf{K}_{23} + \mathbf{K}_{14})\mathbf{d}_{i+1,j} \\
&\quad + \mathbf{K}_{13}\mathbf{d}_{i+1,j+1} + \mathbf{M}_{31}\ddot{\mathbf{d}}_{i-1,j-1} + (\mathbf{M}_{32} + \mathbf{M}_{41})\ddot{\mathbf{d}}_{i-1,j} \\
&\quad + \mathbf{M}_{42}\ddot{\mathbf{d}}_{i-1,j+1} + (\mathbf{M}_{34} + \mathbf{M}_{21})\ddot{\mathbf{d}}_{i,j-1} \\
&\quad + (\mathbf{M}_{11} + \mathbf{M}_{22} + \mathbf{M}_{33} + \mathbf{M}_{44})\ddot{\mathbf{d}}_{i,j} + (\mathbf{M}_{43} + \mathbf{M}_{12})\ddot{\mathbf{d}}_{i,j+1} \\
&\quad + \mathbf{M}_{24}\ddot{\mathbf{d}}_{i+1,j-1} + (\mathbf{M}_{23} + \mathbf{M}_{24})\ddot{\mathbf{d}}_{i+1,j} + \mathbf{M}_{13}\ddot{\mathbf{d}}_{i+1,j+1} = \mathbf{0}
\end{aligned}$$

$$\mathbf{d}_{i+1,j+1} = \mathbf{d}_{i,j} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^n \mathbf{d}_{i,j}$$

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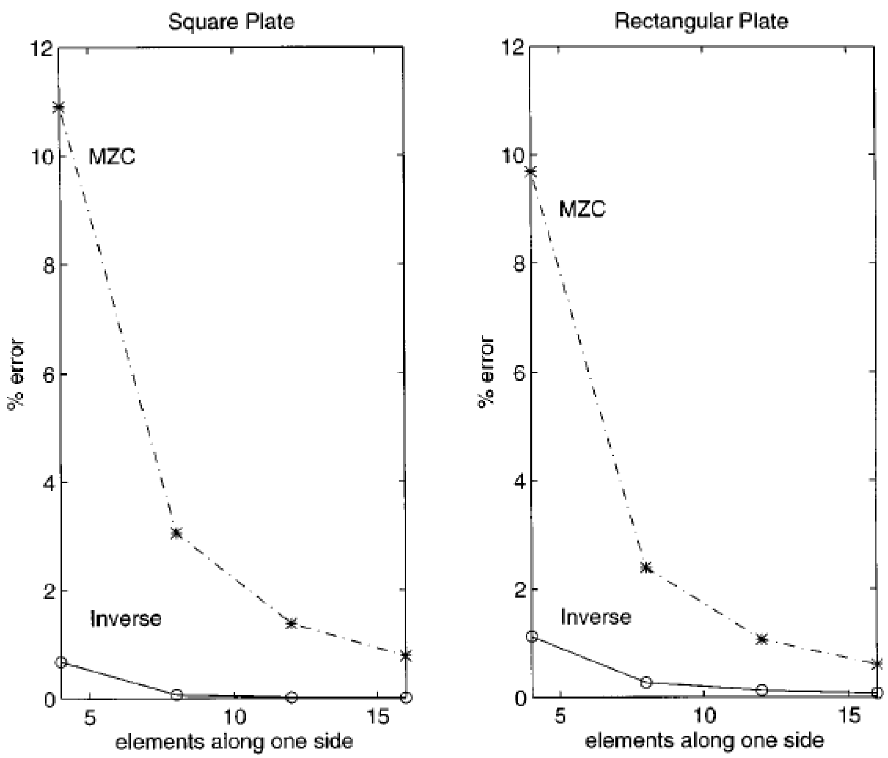

Figure 4. Errors in the estimated centre displacement of a clamped plate

Table I. Dimensionless centre displacement, wD/qL^4 for uniform load q

	MZC model		Inverse model	
Mesh	$p = 1$	$p = 2$	$p = 1$	$p = 2$
4×4	0·001403	0·002778	0·001274	0·002561
8×8	0·001304	0·002593	0·001266	0·002540
12×12	0·001283	0·002560	0·001265	0·002536
16×16	0·001275	0·002548	0·001265	0·002535
Exact[19]	0·001265	0·002533	0·001265	0·002533

PARAMETRIC MODELS AND ERROR ANALYSIS FOR PLATES

Table II. Dimensionless natural frequency of a square fully clamped plate $\omega L^2\sqrt{(\rho/D)}$

Rayleigh–Ritz[20]	MZC model				Inverse model			
	4×4	8×8	12×12	16×16	4×4	8×8	12×12	16×16
35·98	34·31	35·45	35·74	35·84	35·87	35·97	35·98	35·98
73·39	70·03	72·04	72·74	73·01	73·18	73·36	73·39	73·39
73·39	70·03	72·04	72·74	73·01	73·18	73·36	73·39	73·39
108·22	98·06	103·71	106·00	106·92	108·02	108·06	108·18	108·20
131·58	127·58	129·41	130·44	130·90	129·39	131·52	131·57	131·58
132·20	129·62	130·28	131·16	131·58	130·47	132·13	132·19	132·20
165·00	151·01	156·95	160·83	162·52	164·55	164·71	164·92	164·97
165·00	151·01	156·95	160·83	162·52	164·55	164·71	164·92	164·97

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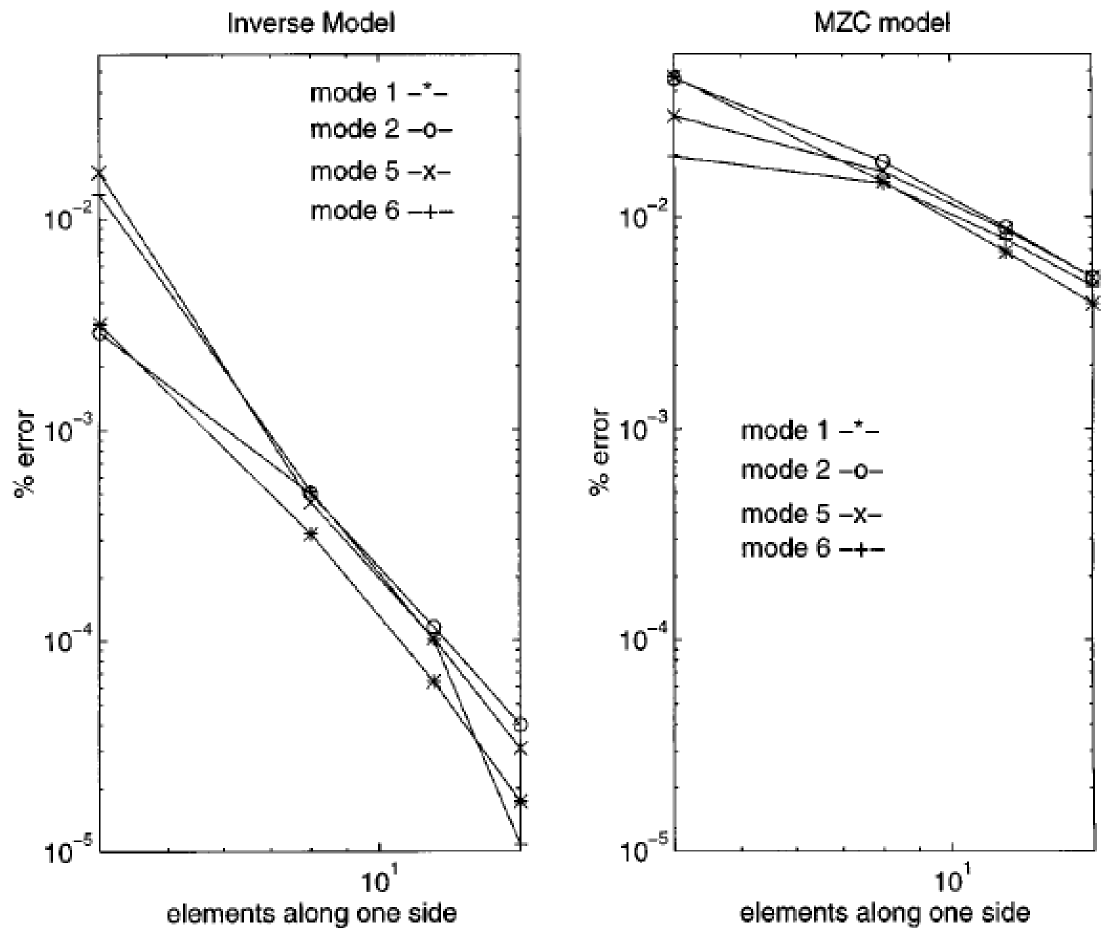

Figure 5. Errors in the estimated eigenvalues of a clamped square plate