Advanced Vibrations

Lecture One

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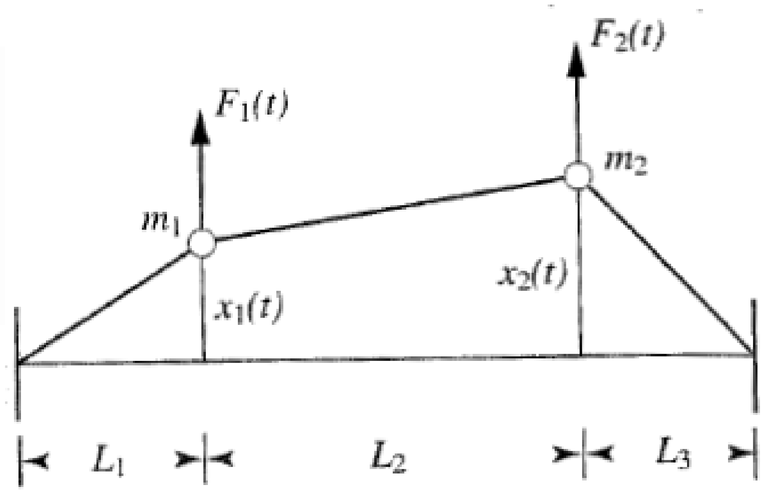
Preliminaries:
Multi-Degree-of-Freedom Systems

1. THE EQUATION OF MOTION
2. FREE VIBRATIONS
3. EIGEN PROBLEM
4. MODE SHAPES
5. RESPONSE TO INITIAL EXCITATIONS
6. COORDINATE TRANSFORMATION
7. ORTHOGONALITY OF MODES, NATURAL COORDINATES
8. BEAT PHENOMENON

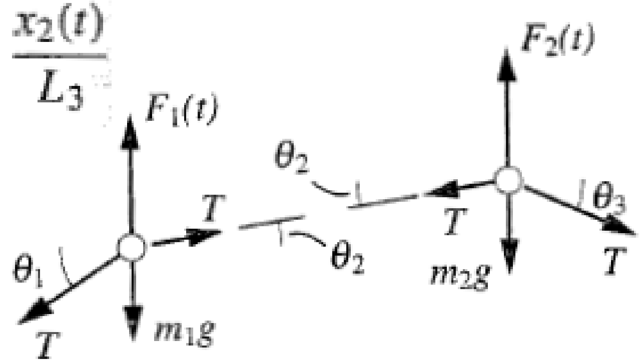
THE EQUATION OF MOTION OF MULTI DEGREE OF FREEIDOM SYSTEMS

$$F_1(t) - m_1 g - T\sin\theta_1(t) + T\sin\theta_2(t) = m_1\frac{d^2x_1(t)}{dt^2}$$

$$F_2(t) - m_2 g - T\sin\theta_2(t) - T\sin\theta_3(t) = m_2\frac{d^2x_2(t)}{dt^2}$$



$$\sin\theta_1(t) \cong \frac{x_1(t)}{L_1},\ \sin\theta_2(t) \cong \frac{x_2(t)-x_1(t)}{L_2},\ \sin\theta_3(t) \cong \frac{x_2(t)}{L_3}$$

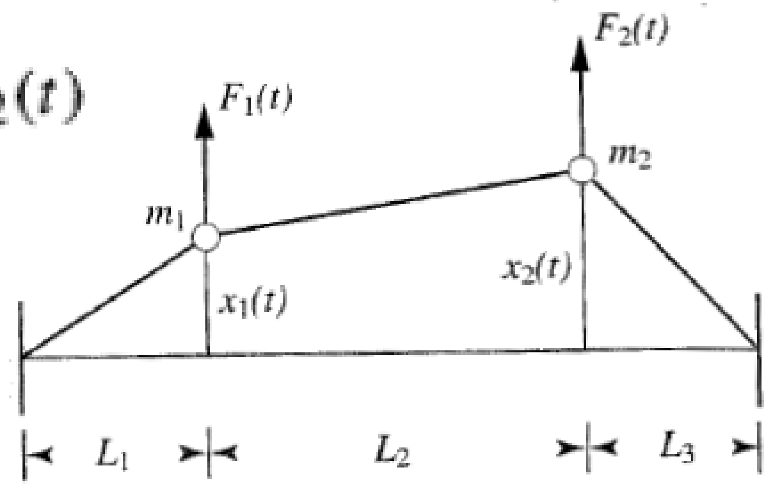


$$m_1\frac{d^2x_1}{dt^2} + \left(\frac{T}{L_1}+\frac{T}{L_2}\right)x_1 - \frac{T}{L_2}x_2 + m_1 g = F_1$$

$$m_2\frac{d^2x_2}{dt^2} - \frac{T}{L_2}x_1 + \left(\frac{T}{L_2}+\frac{T}{L_3}\right)x_2 + m_2 g = F_2$$

THE EQUATION OF MOTION OF MULTI DEGREE OF FREEIDOM SYSTEMS

$$x_1(t) = x_{e1} + \tilde{x}_1(t), \; x_2(t) = x_{e2} + \tilde{x}_2(t)$$

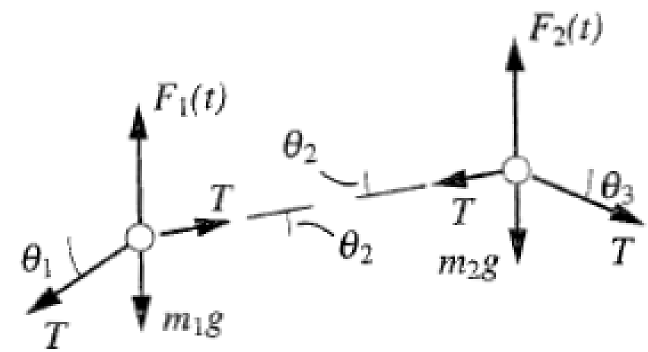


$$\left(\frac{T}{L_1} + \frac{T}{L_2}\right) x_{e1} - \frac{T}{L_2} x_{e2} + m_1 g = 0$$

$$-\frac{T}{L_1} x_{e1} + \left(\frac{T}{L_2} + \frac{T}{L_3}\right) x_{e2} + m_2 g = 0$$

$$m_1 \frac{d^2 \tilde{x}_1}{dt^2} + \left(\frac{T}{L_1} + \frac{T}{L_2}\right) \tilde{x}_1 - \frac{T}{L_2} \tilde{x}_2 = F_1$$

$$m_2 \frac{d^2 \tilde{x}_2}{dt^2} - \frac{T}{L_2} \tilde{x}_1 + \left(\frac{T}{L_2} + \frac{T}{L_3}\right) \tilde{x}_2 = F_2$$

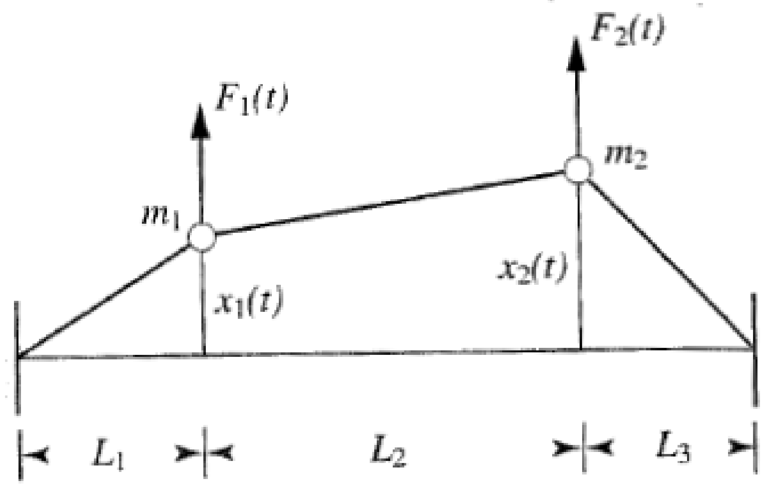


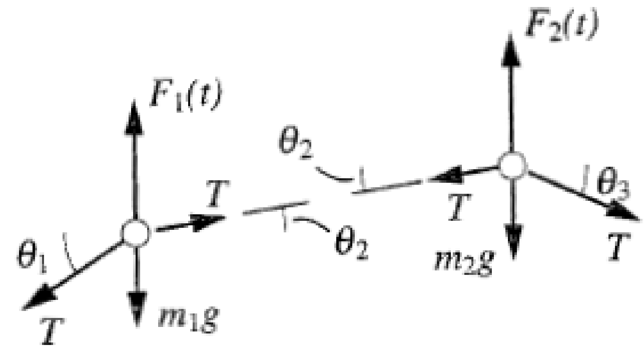
THE EQUATION OF MOTION OF MULTI DEGREE OF FREEIDOM SYSTEMS

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{F}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},$$

$$K = \begin{bmatrix} \dfrac{T}{L_1} + \dfrac{T}{L_2} & -\dfrac{T}{L_2} \\ -\dfrac{T}{L_2} & \dfrac{T}{L_2} + \dfrac{T}{L_3} \end{bmatrix}$$





FREE VIBRATIONS OF UNDAMPED SYSTEMS, NATURAL MODES

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$\mathbf{x}(t) = f(t)\mathbf{u}$$ ← Synchronous motion

$$\ddot{f}(t)M\mathbf{u} + f(t)K\mathbf{u} = \mathbf{0}$$

$$\ddot{f}(t)\mathbf{u}^T M\mathbf{u} + f(t)\mathbf{u}^T K\mathbf{u} = 0$$

$$\frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}} = \lambda$$

$$\ddot{f}(t) + \lambda f(t) = 0$$

$$K\mathbf{u} = \lambda M\mathbf{u}$$

$$\lambda = \omega^2$$

$$\ddot{f}(t) + \omega^2 f(t) = 0$$

$$f(t) = C\cos(\omega t - \phi)$$

EIGEN PROBLEM

$$K\mathbf{u} = \lambda M\mathbf{u}$$

$$(k_{11} - \omega^2 m_1)u_1 + k_{12}u_2 = 0$$

$$k_{12}u_1 + (k_{12} - \omega^2 m_2)u_2 = 0$$

$$\Delta(\omega^2) = \det \begin{bmatrix} k_{11} - \omega^2 m_1 & k_{12} \\ k_{12} & k_{22} - \omega^2 m_2 \end{bmatrix} = 0$$

$$\Delta\omega^2 = m_1 m_2 \left[\omega^4 - \left(\frac{k_{11}}{m_1} + \frac{k_{22}}{m_2} \right) \omega^2 + \frac{k_{11}k_{22} - k_{12}^2}{m_1 m_2} \right] = 0$$

$$\begin{matrix} \omega_1^2 \\ \omega_2^2 \end{matrix} = \frac{1}{2}\left(\frac{k_{11}}{m_1} + \frac{k_{22}}{m_2} \right) \mp \frac{1}{2}\sqrt{\left(\frac{k_{11}}{m_1} + \frac{k_{22}}{m_2} \right)^2 - 4\frac{k_{11}k_{22} - k_{12}^2}{m_1 m_2}}$$

$$f_1(t) = C_1 \cos(\omega_1 t - \phi_1), \quad f_2(t) = C_2 \cos(\omega_2 t - \phi_2)$$

MODE SHAPES

$$K\mathbf{u} = \lambda M\mathbf{u}$$

$$\left.\begin{aligned}(k_{11} - \omega_i^2 m_1)u_{1i} + k_{12}u_{2i} &= 0\\ k_{12}u_{1i} + (k_{22} - \omega_i^2 m_2)u_{2i} &= 0\end{aligned}\right\} i = 1, 2$$

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_1^2 m_2}$$

$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = -\frac{k_{12}}{k_{22} - \omega_2^2 m_2}$$

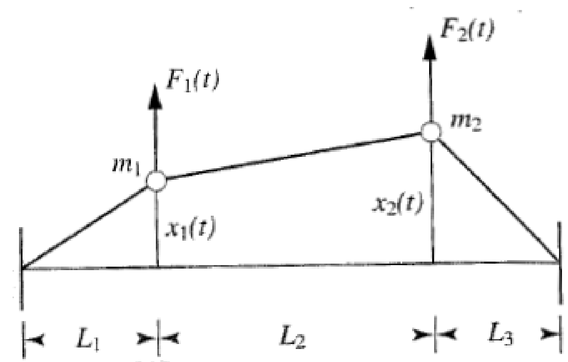
$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

$$\mathbf{x}_1(t) = f_1(t)\mathbf{u}_1 = C_1\mathbf{u}_1\cos(\omega_1 t - \phi_1)$$

$$\mathbf{x}_2(t) = f_2(t)\mathbf{u}_2 = C_2\mathbf{u}_2\cos(\omega_2 t - \phi_2)$$

$$\mathbf{x}(t) = \mathbf{x}_1(t) + \mathbf{x}_2(t) = C_1\cos(\omega_1 t - \phi_1)\mathbf{u}_1 + C_2\cos(\omega_2 t - \phi_2)\mathbf{u}_2$$

EXAMPLE



$$\left.\begin{matrix}\omega_1^2\\ \omega_2^2\end{matrix}\right. = \frac{1}{2}\left(\frac{k_{11}}{m_1}+\frac{k_{22}}{m_2}\right) \mp \frac{1}{2}\sqrt{\left(\frac{k_{11}}{m_1}+\frac{k_{22}}{m_2}\right)^2 - 4\,\frac{k_{11}k_{22}-k_{12}^2}{m_1 m_2}}$$

$$= \frac{1}{2}\left(\frac{2T}{ml}+\frac{3T}{2ml}\right) \mp \frac{1}{2}\sqrt{\left(\frac{2T}{mL}+\frac{3T}{2mL}\right)^2 - 4\left(\frac{2T}{L}\frac{3T}{L}-\frac{T^2}{L^2}\right)\frac{1}{2m^2}}$$

$$= \left[\frac{7}{4} \mp \sqrt{\left(\frac{7}{4}\right)^2 - \frac{5}{2}}\,\right]\frac{T}{mL} = \begin{cases} \dfrac{T}{mL} \\[2ex] \dfrac{5T}{2mL} \end{cases}$$

$$\omega_1 = \sqrt{\frac{T}{mL}},\ \omega_2 = \sqrt{\frac{5T}{2mL}} = 1.581139\sqrt{\frac{T}{mL}}\ \ (\text{rad/s})$$

$$k_{11} = \frac{T}{L_1} + \frac{T}{L_2} = \frac{T}{L} + \frac{T}{L} = \frac{2T}{L}$$

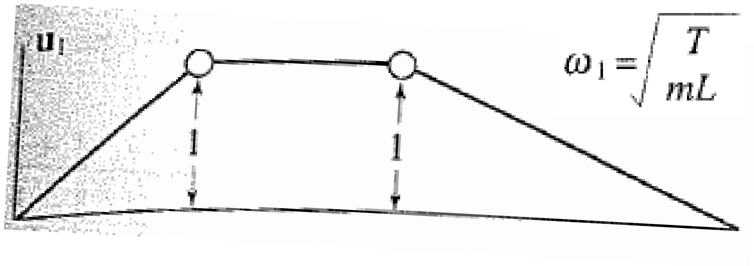
$$k_{22} = \frac{T}{L_2} + \frac{T}{L_3} = \frac{T}{L} + \frac{T}{0.5L} = \frac{3T}{L}$$

$$k_{12} = -\frac{T}{L_2} = -\frac{T}{L}$$

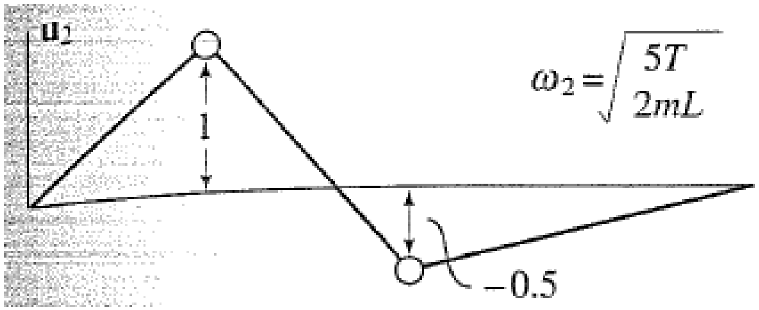
$$m_1 = m,\ m_2 = 2m.$$

EXAMPLE

$$\frac{u_{21}}{u_{11}} = -\frac{k_{11} - \omega_1^2 m_1}{k_{12}} = -\frac{\frac{2T}{L} - \frac{T}{mL}m}{-\frac{T}{L}} = 1$$



$$\frac{u_{22}}{u_{12}} = -\frac{k_{11} - \omega_2^2 m_1}{k_{12}} = -\frac{\frac{2T}{L} - \frac{5T}{2mL}m}{-\frac{T}{L}} = -0.5$$



$$\mathbf{u}_1 = \begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

RESPONSE TO INITIAL EXCITATIONS

$$\mathbf{x}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad \dot{\mathbf{x}}(0) = \mathbf{v}(0) = \begin{bmatrix} v_{10} \\ v_{20} \end{bmatrix}$$

$$x_{10} = u_{11} C_1 \cos\phi_1 + u_{12} C_2 \cos\phi_2$$

$$x_{20} = u_{21} C_1 \cos\phi_1 + u_{22} C_2 \cos\phi_2$$

$$v_{10} = \omega_1 u_{11} C_1 \sin\phi_1 + \omega_2 u_{12} C_2 \sin\phi_2$$

$$v_{20} = \omega_1 u_{21} C_1 \sin\phi_1 + \omega_2 u_{22} C_2 \sin\phi_2$$

$$C_1 \cos\phi_1 = \frac{u_{22} x_{10} - u_{12} x_{20}}{|U|}, \quad C_2 \cos\phi_2 = \frac{u_{11} x_{20} - u_{21} x_{10}}{|U|}$$

$$C_1 \sin\phi_1 = \frac{u_{22} v_{10} - u_{12} v_{20}}{\omega_1 |U|}, \quad C_2 \sin\phi_2 = \frac{u_{11} v_{20} - u_{21} v_{10}}{\omega_2 |U|}$$

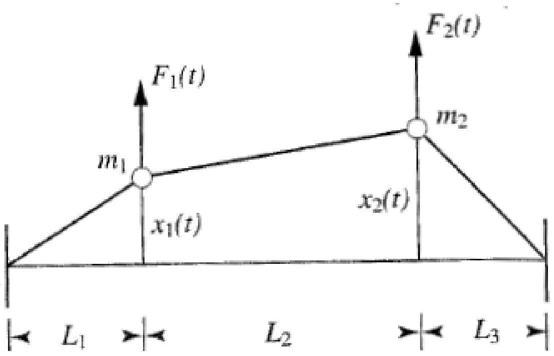
where $|U|$ is the determinant of the *modal matrix* U,

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

EXAMPLE

X_1(0)= 1.2 cm. The other initial conditions are zero.

$$\omega_1 = \sqrt{\frac{T}{mL}}, \quad \omega_2 = 1.581139\sqrt{\frac{T}{mL}} \quad (\text{rad/s})$$



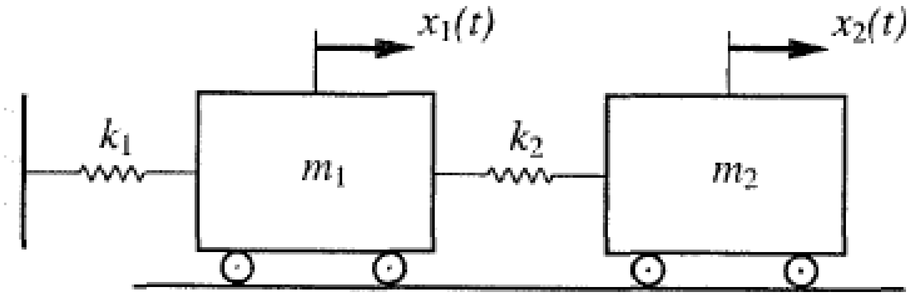
$$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}$$

$$|U| = u_{11}u_{22} - u_{12}u_{21} = -1.5$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{-1.5}\left\{(-0.5)\times 1.2\cos\sqrt{\frac{T}{mL}}t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\times 1.2\cos 1.581139\sqrt{\frac{T}{mL}}t\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 0.4\cos\sqrt{\frac{T}{mL}}t + 0.8\cos 1.581139\sqrt{\frac{T}{mL}}t \\ 0.4\cos\sqrt{\frac{T}{mL}}t - 0.4\cos 1.581139\sqrt{\frac{T}{mL}}t \end{bmatrix} (\text{cm})$$

COORDINATE TRANSFORMATION, COUPLING



$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$x_1(t) = z_1(t), \quad x_2(t) = z_1(t) + z_2(t)$$

$$\mathbf{x}(t) = T\mathbf{z}(t) \qquad M'\ddot{\mathbf{z}}(t) + K'\mathbf{z}(t) = \mathbf{0}$$

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{bmatrix}$$

$$K' = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

ORTHOGONALITY OF MODES, NATURAL COORDINATES

$$K\mathbf{u}_1 = \omega_1^2 M\mathbf{u}_1$$

$$K\mathbf{u}_2 = \omega_2^2 M\mathbf{u}_2$$

$$\mathbf{x}(t) = q_1(t)\mathbf{u}_1 + q_2(t)\mathbf{u}_2$$

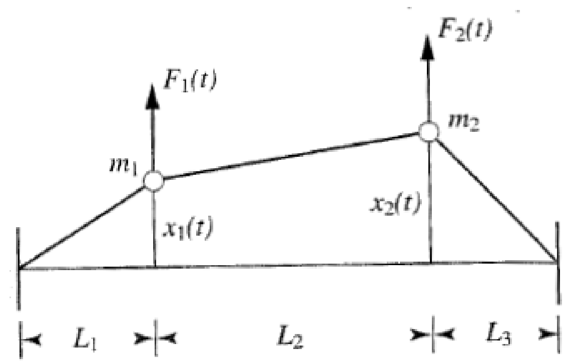
$$\ddot{q}_1(t) + \omega_1^2 q_1(t) = 0$$

$$\ddot{q}_2(t) + \omega_2^2 q_2(t) = 0$$

$$q_1(t) = C_1\cos(\omega_1 t - \phi_1),\ \ q_2(t) = C_2\cos(\omega_2 t - \phi_2)$$

$$\mathbf{x}(t) = C_1\cos(\omega_1 t - \phi_1)\mathbf{u}_1 + C_2\cos(\omega_2 t - \phi_2)\mathbf{u}_2$$

EXAMPLE



$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \mathbf{x}(0) = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix}, \dot{\mathbf{x}}(0) = \mathbf{0}$$

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} = \frac{T}{L}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\omega_1 = \sqrt{\frac{T}{mL}}, \ \omega_2 = \sqrt{\frac{5T}{2mL}} = 1.581139\sqrt{\frac{T}{mL}} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

Introducing the linear transformation:

$$\ddot{q}_1(t) + \sqrt{\frac{T}{mL}}q_1(t) = 0$$

$$\mathbf{x}(t) = q_1(t)\mathbf{u}_1 + q_2(t)\mathbf{u}_2 = q_1(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + q_2(t)\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$\ddot{q}_2(t) + \sqrt{\frac{5T}{2mL}}q_2(t) = 0$$

$$\mathbf{x}(0) = \begin{bmatrix} 1.2 \\ 0 \end{bmatrix} = q_1(0)\mathbf{u}_1 + q_2(0)\mathbf{u}_2 = q_1(0)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + q_2(0)\begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \qquad q_1(0) = 0.4, \ q_2(0) = 0.8$$

$$q_1(t) = q_1(0)\cos\omega_1 t = 0.4\cos\sqrt{\frac{T}{mL}}t \qquad q_2(t) = q_2(0)\cos\omega_2 t = 0.8\cos 1.581139\sqrt{\frac{T}{mL}}t$$

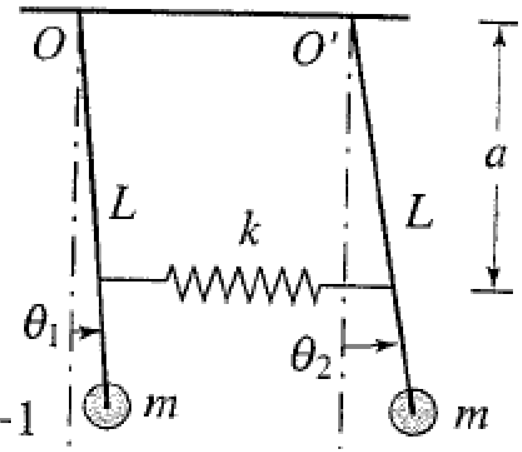
$$\mathbf{x}(t) = 0.4\cos\sqrt{\frac{T}{mL}}t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.8\cos 1.581139\sqrt{\frac{T}{mL}}t\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

BEAT PHENOMENON

$$\begin{bmatrix} mL^2 & 0 \\ 0 & mL^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} mgL+ka^2 & -ka^2 \\ -ka^2 & mgL+ka^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

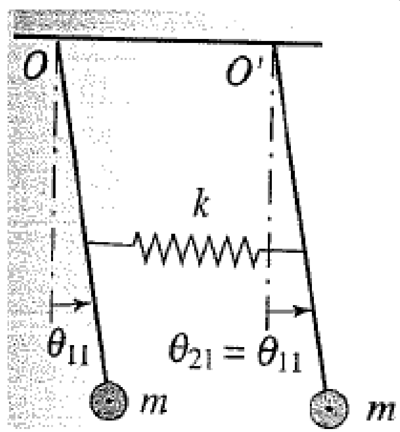
$$\det\begin{bmatrix} mgL+ka^2-\omega^2 mL^2 & -ka^2 \\ -ka^2 & mgL+ka^2-\omega^2 mL^2 \end{bmatrix}$$

$$= (mgL+ka^2-\omega^2 mL^2)^2 - (ka^2)^2 = 0$$

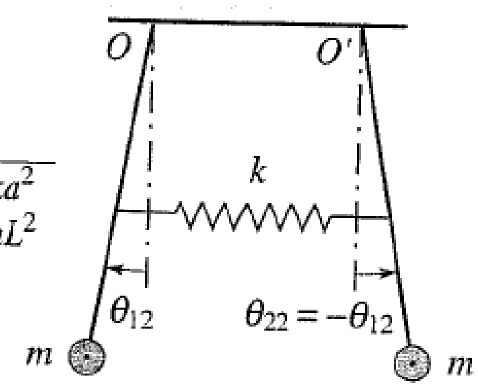


$$\omega_1 = \sqrt{\frac{g}{L}},\ \omega_2 = \sqrt{\frac{g}{L} + 2\frac{k}{m}\frac{a^2}{L^2}} \quad \frac{\Theta_{21}}{\Theta_{11}} = 1,\ \frac{\Theta_{22}}{\Theta_{12}} = -1$$

$$\omega_1 = \sqrt{\frac{g}{L}}$$



$$\omega_2 = \sqrt{\frac{g}{L} + 2\frac{ka^2}{mL^2}}$$



BEAT PHENOMENON

Then, considering the initial conditions:

$$\theta_1(0) = \theta_0,\ \theta_2(0) = 0,\ \dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$$

$$\theta_1(t) = \tfrac{1}{2}\theta_0 \cos\omega_1 t + \tfrac{1}{2}\theta_0 \cos\omega_2 t = \theta_0 \cos\frac{\omega_2 - \omega_1}{2} t \cos\frac{\omega_2 + \omega_1}{2} t$$

$$\theta_2(t) = \tfrac{1}{2}\theta_0 \cos\omega_1 t - \tfrac{1}{2}\theta_0 \cos\omega_2 t = \theta_0 \sin\frac{\omega_2 - \omega_1}{2} t \sin\frac{\omega_2 + \omega_1}{2} t$$

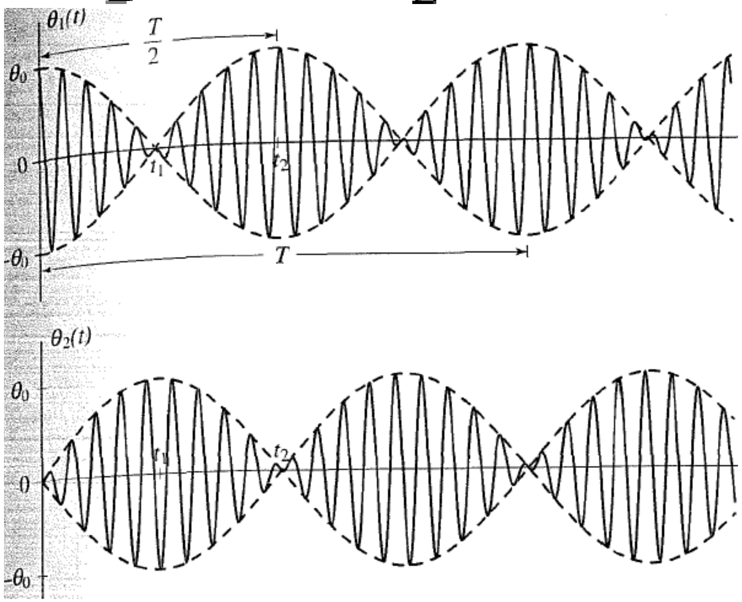
$k \ll mgL/a^2$

$$\frac{\omega_B}{2} = \frac{\omega_2 - \omega_1}{2} \cong \frac{1}{2}\frac{k}{m}\frac{a^2}{\sqrt{gL^3}},$$

$$\omega_{\text{ave}} = \frac{\omega_2 + \omega_1}{2} \cong \sqrt{\frac{g}{L}} + \frac{1}{2}\frac{k}{m}\frac{a^2}{\sqrt{gL^3}}$$

$$\theta_1(t) \cong \theta_0 \cos\tfrac{1}{2}\omega_B t \cos\omega_{\text{ave}} t,$$

$$\theta_2(t) \cong \theta_0 \sin\tfrac{1}{2}\omega_B t \sin\omega_{\text{ave}} t$$



WE COVERED:
Multi-Degree-of-Freedom Systems

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Advanced Vibrations

Lecture Two

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UMASS LOWELL MODAL ANALYSIS and CONTROLS LABORATORY - Pete Avitabile and Fabio Piergentili

Preliminaries:
Multi-Degree-of-Freedom Systems

1. RESPONSE TO HARMONIC EXCITATIONS
2. UNDAMPED VIBRATION ABSORBERS
3. RESPONSE TO NONPERODIC EXCITATIONS

RESPONSE TO HARMONIC EXCITATIONS

$$M\ddot{\mathbf{x}}(t) + C\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t)$$

$$\mathbf{F}(t) = \mathbf{F}e^{i\omega t} \qquad \mathbf{x}(t) = \mathbf{X}(i\omega)e^{i\omega t}$$

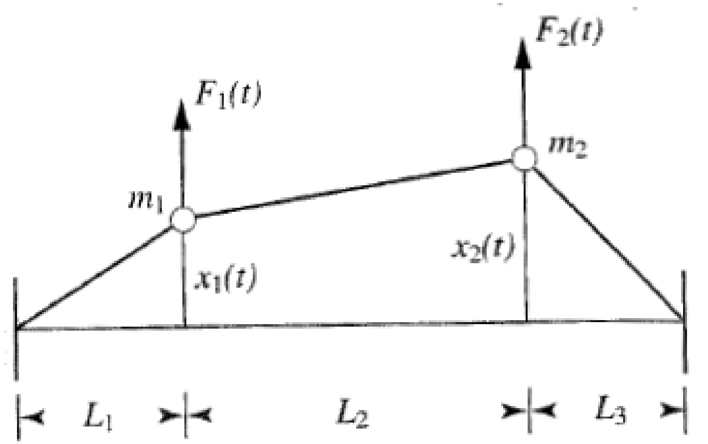
$$\mathbf{Z}(i\omega)\mathbf{X}(i\omega) = \mathbf{F}$$

$$Z(i\omega) = -\omega^2 M + i\omega C + K$$

$$\mathbf{X}(i\omega) = \mathbf{Z}^{-1}(i\omega)\mathbf{F}$$

EXAMPLE:

Plot the frequency response curves when F_2=0.



$$Z^{-1}(i\omega) = \frac{1}{|Z(i\omega)|}\begin{bmatrix} z_{22}(i\omega) & -z_{12}(i\omega) \\ -z_{12}(i\omega) & z_{11}(i\omega) \end{bmatrix}$$

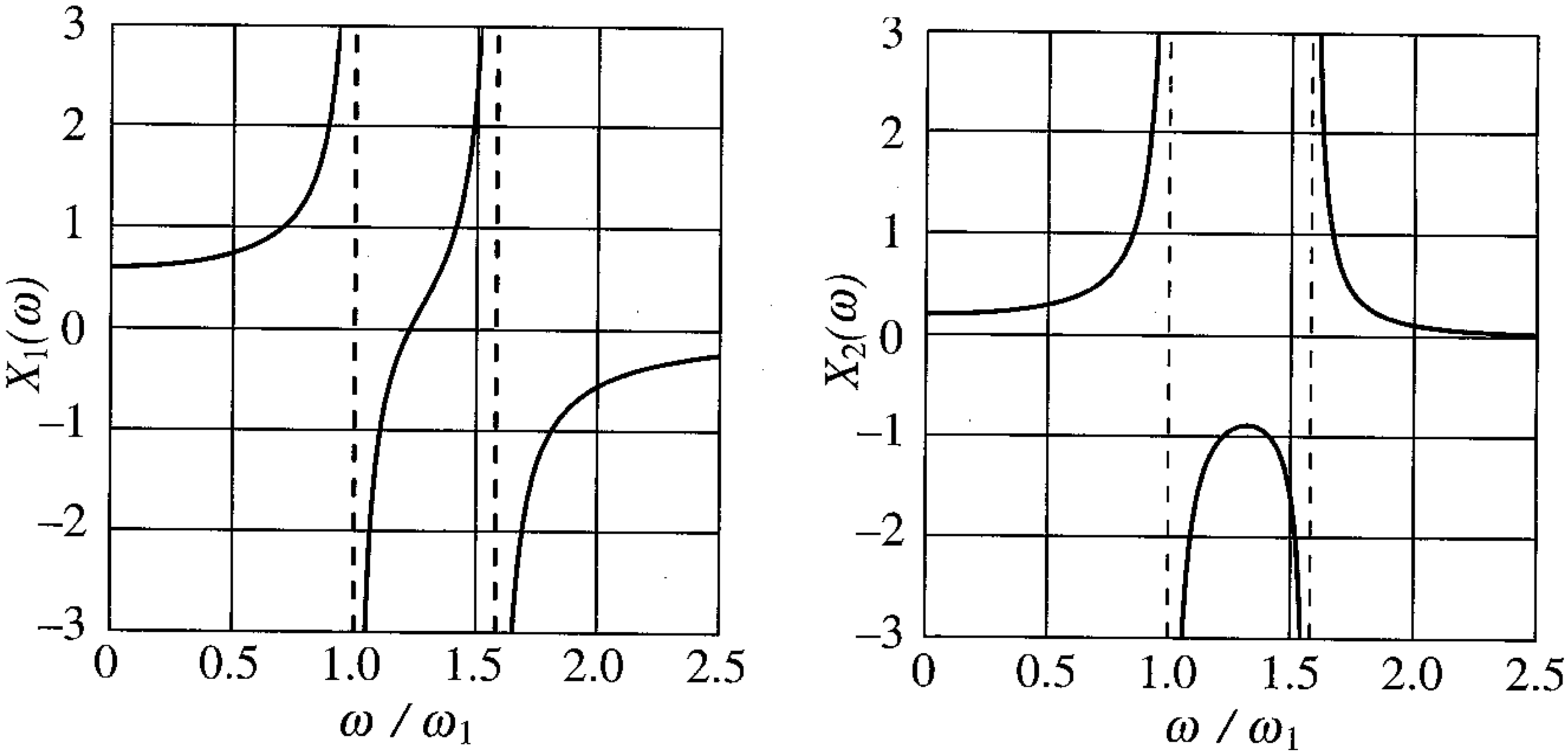
$$X_1(\omega) = \frac{(k_{22}-\omega^2 m_2)F_1 - k_{12}F_2}{(k_{11}-\omega^2 m_1)(k_{22}-\omega^2 m_2) - k_{12}^2}$$

$$T/L = k$$

$$X_2(\omega) = \frac{-k_{12}F_1 + (k_{11}-\omega^2 m_1)F_2}{(k_{11}-\omega^2 m_1)(k_{22}-\omega^2 m_2) - k_{12}^2}$$

EXAMPLE:

$$\omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{5}{2}\frac{k}{m}$$



$$X_1(\omega) = \frac{(3k - 2m\omega^2)F_1}{2m^2\omega^4 - 7mk\omega^2 + 5k^2}$$

$$X_2(\omega) = \frac{kF_1}{2m^2\omega^4 - 7mk\omega^2 + 5k^2}$$

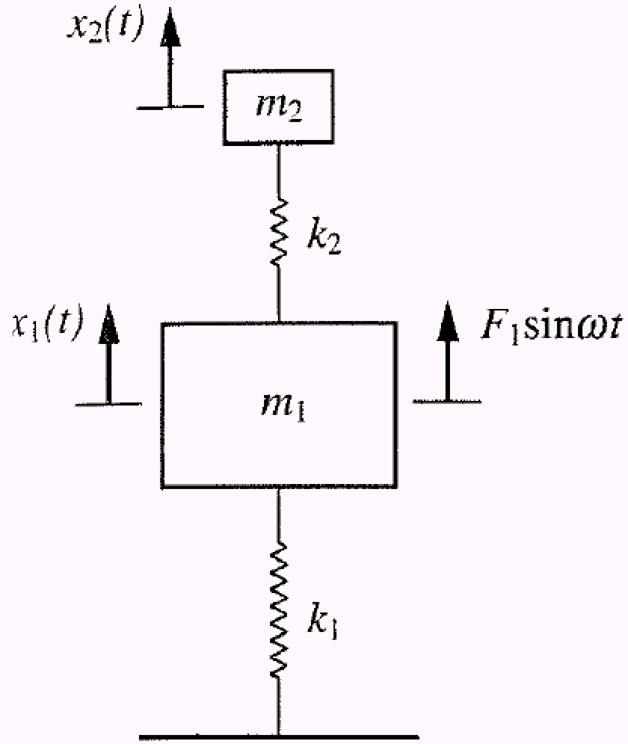
UNDAMPED VIBRATION ABSORBERS

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1 \sin\omega t$$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

$$x_1(t) = X_1 \sin\omega t, \quad x_2(t) = X_2 \sin\omega t$$

$$\begin{bmatrix} k_1 + k_2 - \omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \omega^2 m_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

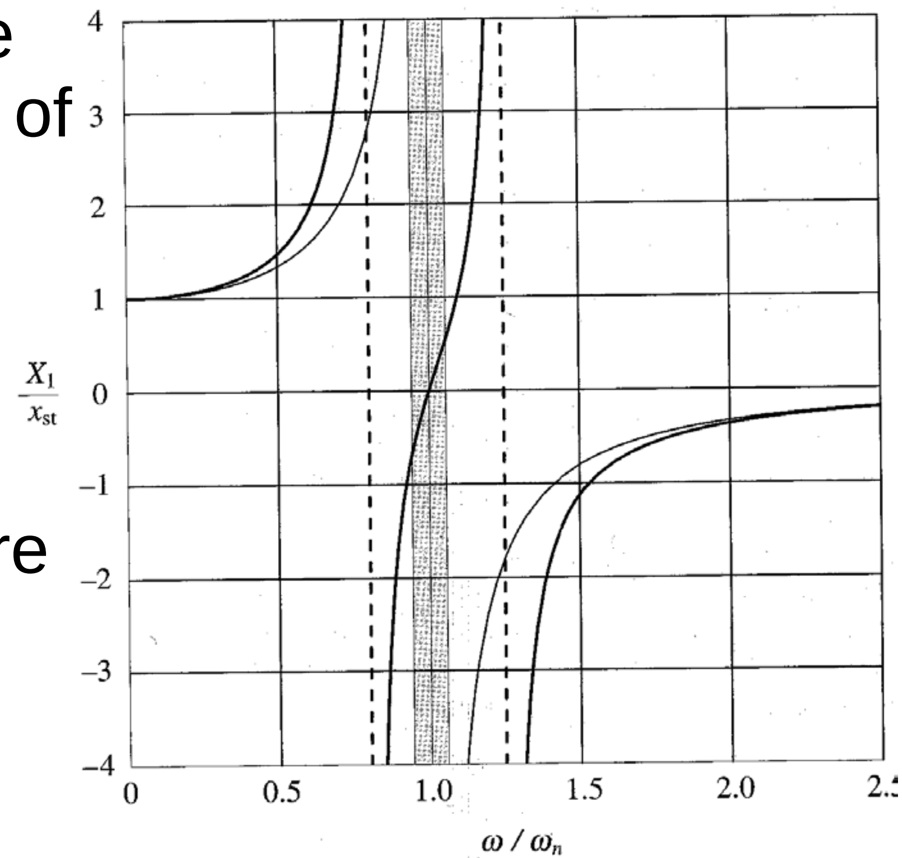


$$X_1 = \frac{(k_2 - \omega^2 m_2)F_1}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2}$$

$$X_2 = \frac{k_2 F_1}{(k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2}$$

UNDAMPED VIBRATION ABSORBERS

$\mu = 0.2$ and $\omega_n = \omega_a$.



The shaded area indicates the region which the performance of the absorber can be regarded as satisfactory.

One disadvantage of the vibration absorber is that two new resonance frequencies are created.

$$\omega_n = \sqrt{k_1/m_1} \qquad x_{st} = F_1/k_1$$

$$\omega_a = \sqrt{k_2/m_2} \qquad \mu = m_2/m_1$$

RESPONSE TO NON-PERIODIC EXCITATIONS

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{F}(t)$$

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}, \; K = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}$$

$$\mathbf{x}(t) = \eta_1(t)\mathbf{u}_1 + \eta_2(t)\mathbf{u}_2$$

Modal Coordinates

$$m'_{11}\ddot{\eta}_1(t) + m'_{11}\omega_1^2\eta_1(t) = N_1(t)$$

$$m'_{22}\ddot{\eta}_2(t) + m'_{22}\omega_2^2\eta_2(t) = N_2(t)$$

Modal Forces

$$N_1(t) = \mathbf{u}_1^T\mathbf{F}(t), \; N_2(t) = \mathbf{u}_2^T\mathbf{F}(t)$$

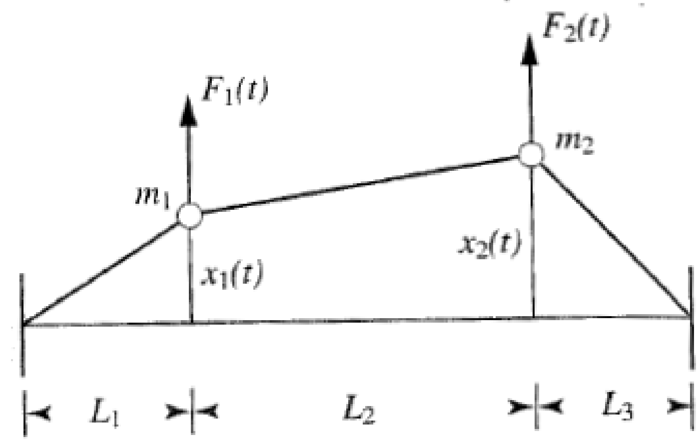
RESPONSE TO NON-PERIODIC EXCITATIONS

$$\eta_1(t) = \int_0^t N_1(t-\tau)g_1(\tau)d\tau = \frac{1}{m'_{11}\omega_1}\int_0^t N_1(t-\tau)\sin\omega_1\tau d\tau$$

$$\eta_2(t) = \int_0^t N_2(t-\tau)g_2(\tau)d\tau = \frac{1}{m'_{22}\omega_2}\int_0^t N_2(t-\tau)\sin\omega_2\tau d\tau$$

$$\mathbf{x}(t) = \eta_1(t)\mathbf{u}_1 + \eta_2(t)\mathbf{u}_2$$

EXAMPLE:



$$F_2(t) = F_0[u(t) - u(t-a)]$$

$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad K = \frac{T}{L}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$\omega_1 = \sqrt{\frac{T}{mL}}, \quad \omega_2 = \sqrt{\frac{5T}{2mL}} \qquad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$N_1(t) = \mathbf{u}_1^T \mathbf{F}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ F_2(t) \end{bmatrix} = F_2(t)$$

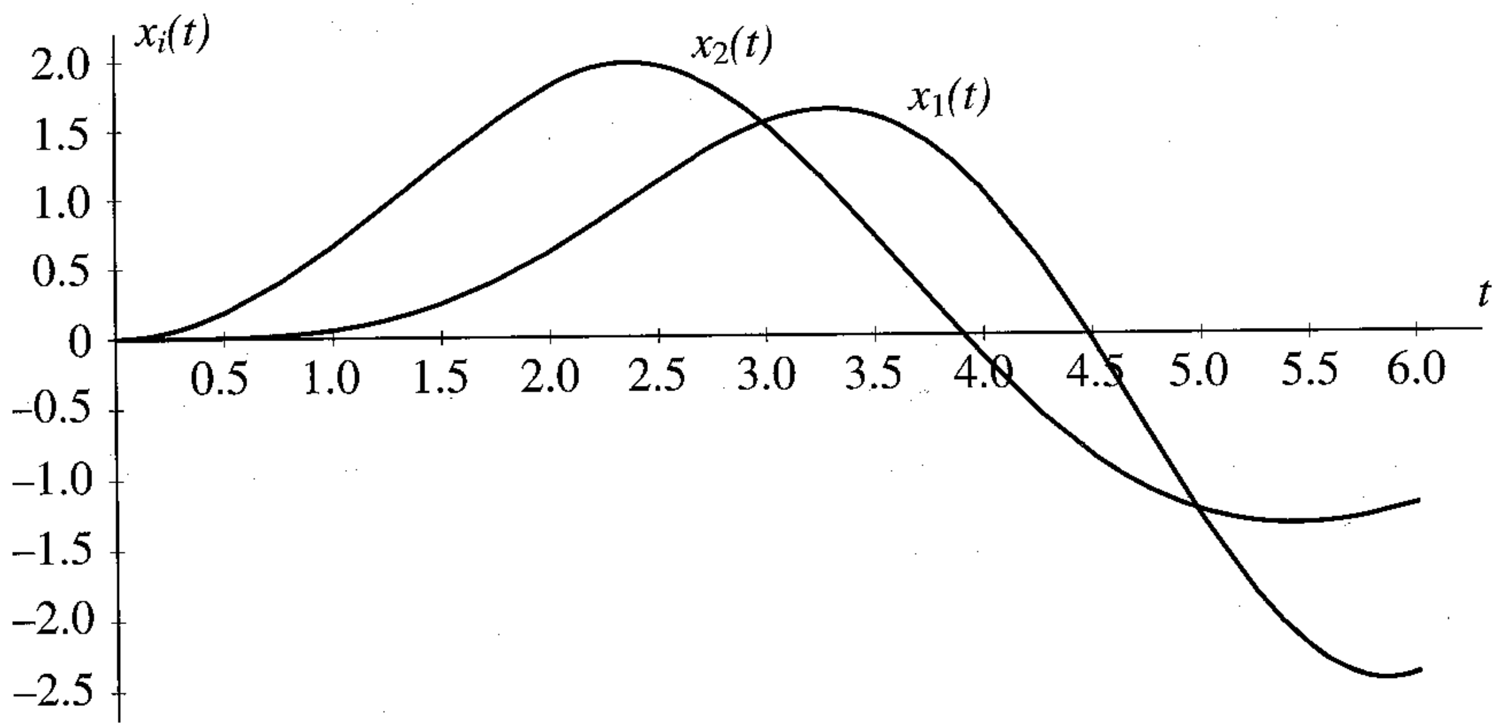
$$N_2(t) = \mathbf{u}_2^T \mathbf{F}(t) = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}^T \begin{bmatrix} 0 \\ F_2(t) \end{bmatrix} = -0.5F_2(t)$$

EXAMPLE:

$$\eta_1(t) = \frac{1}{m'_{11}\omega_1}\int_0^t N_1(t-\tau)\sin\omega_1\tau\,\cancel{u}(\tau)\,d\tau = \frac{1}{m'_{11}\omega_1}\int_0^t N_1(t-\tau)\sin\omega_1\tau\,d\tau$$

$$\eta_2(t) = \frac{1}{m'_{22}\omega_2}\int_0^t N_2(t-\tau)\sin\omega_2\tau\,\cancel{u}(\tau)\,d\tau = \frac{1}{m'_{22}\omega_2}\int_0^t N_2(t-\tau)\sin\omega_2\tau\,d\tau$$

$T/L = 1\ \mathrm{N/m}$,
$m = 1\ \mathrm{kg}$,
$a = 2\ \mathrm{s}$



Advanced Vibrations

Lecture Three

Elements of Analytical Dynamics

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Elements of Analytical Dynamics

Newton's laws were formulated for a single particle

- Can be extended to systems of particles.
- The equations of motion are expressed in terms of physical coordinates vector and force vector.
  - For this reason, *Newtonian mechanics* is often referred to as *vectorial mechanics.*

The drawback is that it requires one free-body diagram for each of the masses,

- Necessitating the inclusion of reaction forces and interacting forces.
- These reaction and constraint forces play the role of unknowns, which makes it necessary to work with a surplus of equations of motion, one additional equation for every unknown force.

Elements of Analytical Dynamics

Analytical mechanics, or *analytical dynamics*, considers the system as a whole:

- Not separate individual components,
- This excludes the reaction and constraint forces automatically.

This approach, due to Lagrange, permits the formulation of problems of dynamics in terms of:

- two scalar functions the kinetic energy and the potential energy, and
- an infinitesimal expression, the virtual work performed by the nonconservative forces.

Elements of Analytical Dynamics

In analytical mechanics the equations of motion are formulated in terms of generalized coordinates and generalized forces:

- Not necessarily physical coordinates and forces.
- The formulation is independent of any special system of coordinates.

The development of analytical mechanics required the introduction of the concept of virtual displacements,

- led to the development of the calculus of variations.
- For this reason, analytical mechanics is often referred to as the *variational approach to mechanics.*

6 Elements of Analytical Dynamics

6.1 DOF and Generalized Coordinates

6.2 The Principle of Virtual Work

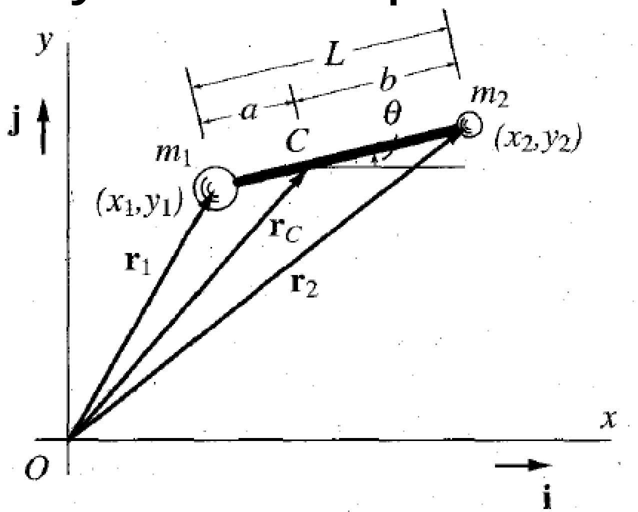
6.3 The Principle of D'Alembert

6.4 The Extended Hamilton's Principle

6.5 Lagrange's Equations

6.1 DEGREES OF FREEDOM AND GENERALIZED COORDINATES

A source of possible difficulties in using Newton's equations is use of physical coordinates, which may not always be independent.



$$\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j},\ \mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j}$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = L^2$$

Independent coordinates

$$\mathbf{r}_C = \mathbf{r}_C(x_C, y_C) \text{ and } \theta.$$

$$\mathbf{r}_1 = \mathbf{r}_C - a(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}),$$
$$\mathbf{r}_2 = \mathbf{r}_C + b(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$a = \frac{m_2 L}{m_1 + m_2},\quad b = \frac{m_1 L}{m_1 + m_2}$$

The generalized coordinates are not unique

6.2 THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work, due to ***Johann Bernoulli***, is basically a statement of the static equilibrium of a mechanical system.

We consider a system of N particles and define the *virtual displacements*, as *infinitesimal changes* in the coordinates.

The virtual displacements must be *consistent with the system constraints,* but are otherwise *arbitrary.*

The virtual displacements, being infinitesimal, *obey the rules of differential calculus.*

THE PRINCIPLE OF VIRTUAL WORK

$$\mathbf{R}_i = \mathbf{F}_i + \mathbf{f}_i = \mathbf{0}, \; i = 1, 2, \ldots, N$$

resultant force on each particle *applied force* *constraint force*

$$\overline{\delta W}_i = \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0, \; i = 1, 2, \ldots, N$$

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{R}_i \cdot \delta \mathbf{r}_i = 0$$

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

The virtual work performed by the constraint forces is zero

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

THE PRINCIPLE OF VIRTUAL WORK

When $\mathbf{r}_i$ are independent,

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \longrightarrow \mathbf{F}_i = \mathbf{0},\ i = 1, 2, \ldots, N$$

If not to switch to a set of generalized coordinates: $\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n),\ i = 1, 2, \ldots, N$

$$\delta \mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\delta q_2 + \ldots + \frac{\partial \mathbf{r}_i}{\partial q_n}\delta q_n = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k,\ i = 1, 2, \ldots, N$$

$$\overline{\delta W} = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i \cdot \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\left(\sum_{i=1}^{N} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\right)\delta q_k = \sum_{k=1}^{n} Q_k \delta q_k = 0$$

$$\longrightarrow Q_k = 0,\ k = 1, 2, \ldots, n$$

Generalized forces

THE PRINCIPLE OF D'ALEMBERT

The virtual work principle can be extended to dynamics, in which form it is known as d'Alembert's principle.

$$\mathbf{F}_i + \mathbf{f}_i - m_i \ddot{\mathbf{r}}_i = \mathbf{0}, \ i = 1, 2, \ldots, N$$

$$(\mathbf{F}_i + \mathbf{f}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0, \ i = 1, 2, \ldots, N$$

$$\sum_{i=1}^{N} (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$$

Lagrange version of d'Alembertls principle

THE EXTENDED HAMILTON'S PRINCIPLE

$$\sum_{i=1}^{N}(\mathbf{F}_i - m_i\ddot{\mathbf{r}}_i)\cdot\delta\mathbf{r}_i = 0$$

$$\sum_{i=1}^{N}\mathbf{F}_i\cdot\delta\mathbf{r}_i = \overline{\delta W}$$

The virtual work of all the applied forces,

$$\frac{d}{dt}(m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}) = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + m_i\dot{\mathbf{r}}_i\cdot\delta\dot{\mathbf{r}}_i = m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + \delta(\tfrac{1}{2}m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i)$$

$$= m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i + \delta T_i$$

The kinetic energy of particle m_i

$$-\int_{t_1}^{t_2} m_i\ddot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\,dt = \int_{t_1}^{t_2}\delta T_i\,dt - \int_{t_1}^{t_2}\frac{d}{dt}(m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i)\,dt$$

$$= \int_{t_1}^{t_2}\delta T_i\,dt - m_i\dot{\mathbf{r}}_i\cdot\delta\mathbf{r}_i\Big|_{t_1}^{t_2}$$

THE EXTENDED HAMILTON'S PRINCIPLE

It is convenient to choose $\delta \mathbf{r}_i = \mathbf{0}$ at $t = t_1$ and $t = t_2$.

$$-\int_{t_1}^{t_2} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt = \int_{t_1}^{t_2} \delta T_i dt, \quad \delta \mathbf{r}_i = \mathbf{0}, \quad t = t_1, t_2; \quad i = 1, 2, \ldots, N$$

$$-\int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i dt = \int_{t_1}^{t_2} \delta T dt, \quad \delta \mathbf{r}_i = \mathbf{0}, \quad i = 1, 2, \ldots, N; \quad t = t_1, t_2$$

$$\int_{t_1}^{t_2} (\delta T + \overline{\delta W}) dt = 0, \quad \delta \mathbf{r}_i = \mathbf{0}, \quad i = 1, 2, \ldots, N; \quad t = t_1, t_2$$

Extended Hamilton's principle

THE EXTENDED HAMILTON'S PRINCIPLE

$$\int_{t_1}^{t_2} (\delta T + \overline{\delta W})dt = 0,\ \delta \mathbf{r}_i = \mathbf{0},\ i = 1, 2, \ldots, N;\ t = t_1, t_2$$

$$\overline{\delta W} = \overline{\delta W_c} + \overline{\delta W}_{nc} = -\delta V + \overline{\delta W}_{nc}$$

where V is the potential energy

$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W}_{nc})dt = 0,\ \delta \mathbf{r}_i = \mathbf{0},\ i = 1, 2, \ldots, N;\ t = t_1, t_2$$

Or in terms of the independent generalized coordinates

$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W}_{nc})dt = 0,\ \delta q_k = 0,\ k = 1, 2, \ldots, n;\ t = t_1, t_2$$

A Note:

$$\overline{dW} = \mathbf{F}\cdot d\mathbf{r}$$

$$\overline{dW} = m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}dt = m\frac{d\dot{\mathbf{r}}}{dt}\cdot\dot{\mathbf{r}}dt = m\dot{\mathbf{r}}\cdot d\dot{\mathbf{r}} = d\left(\frac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\right)$$

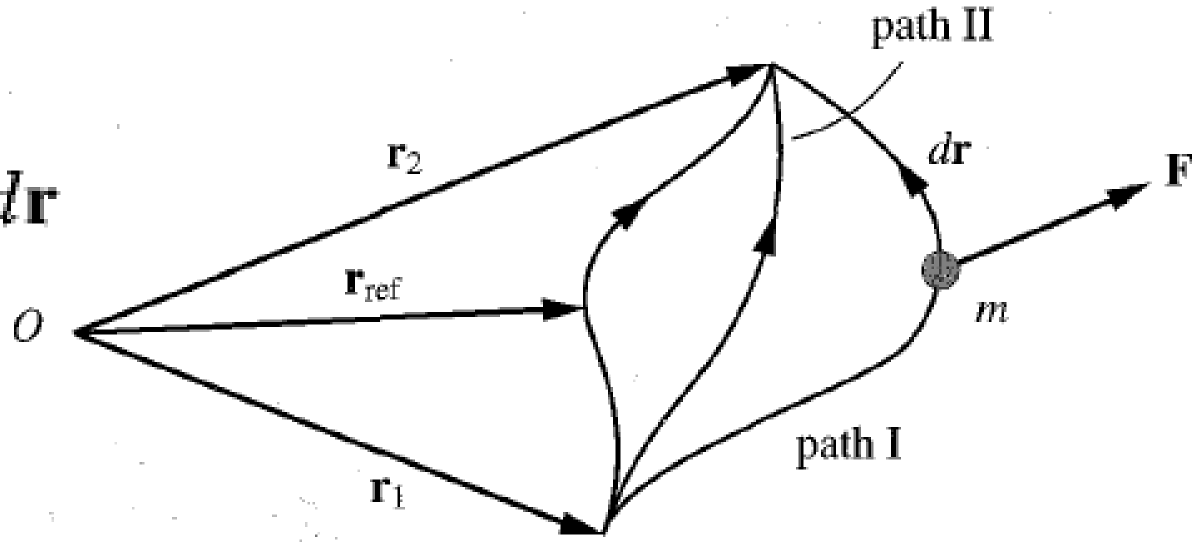
$$T = \frac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} \qquad \overline{dW} = dT$$

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2}\mathbf{F}\cdot d\mathbf{r} = \int_{T_1}^{T_2} dT = T_2 - T_1$$

The work performed by the force F in moving the particle m from position $\mathbf{r}_1$ *to position* $\mathbf{r}_2$ *is responsible for a change in the kinetic energy from* T_1 *to* T_2*.*

A Note (continued):

$$V(\mathbf{r}) = \int_{\mathbf{r}}^{\mathbf{r}_{ref}} \mathbf{F}_c \cdot d\mathbf{r}$$



$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_{ref}} \mathbf{F}_c \cdot d\mathbf{r} + \int_{\mathbf{r}_{ref}}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_{ref}} \mathbf{F}_c \cdot d\mathbf{r} - \int_{\mathbf{r}_2}^{\mathbf{r}_{ref}} \mathbf{F}_c \cdot d\mathbf{r}$$

$$= V(\mathbf{r}_1) - V(\mathbf{r}_2) = -(V_2 - V_1)$$

The work performed by conservative forces in moving a particle from r_1 to $\mathbf{r}_2$ is equal to the negative of the change in the potential energy from $\mathbf{V_1}$ to $\mathbf{V_2}$

A Note (continued):

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_c \cdot d\mathbf{r} + \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc} \cdot d\mathbf{r}$$

$$T_2 - T_1 = -(V_2 - V_1) + \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc} \cdot d\mathbf{r}$$

$$E = T + V \qquad \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F}_{nc} \cdot d\mathbf{r} = E_2 - E_1$$

$$\mathbf{F}_{nc} \cdot d\mathbf{r} = dE$$

$$\mathbf{F}_{nc} \cdot \dot{\mathbf{r}} = \dot{E}$$

6 Elements of Analytical Dynamics

6.1 DOF and Generalized Coordinates

6.2 The Principle of Virtual Work

6.3 The Principle of D'Alembert

6.4 The Extended Hamilton's Principle

6.5 Lagrange's Equations

Advanced Vibrations

Lecture Four:
LAGRANGE'S EQUATIONS

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THE EXTENDED HAMILTON'S PRINCIPLE

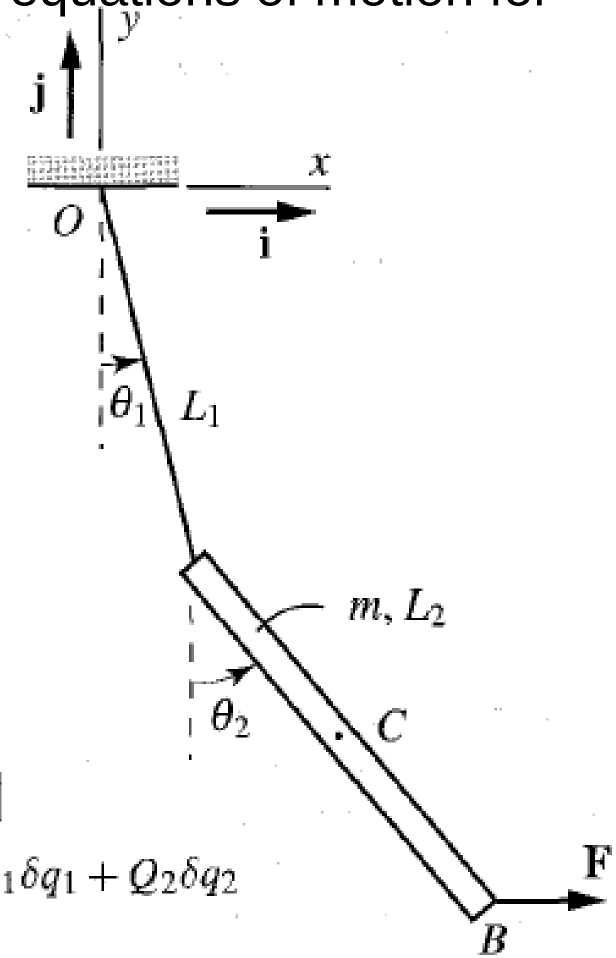
$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W}_{nc})dt = 0,$$

$$\int_{t_1}^{t_2} \delta L\, dt = 0, \quad \delta q_k = 0, \qquad L = T - V$$

Hamilton's principle *Lagrangian*

THE EXTENDED HAMILTON'S PRINCIPLE

Use the extended Hamilton's principle to derive the equations of motion for the two-degree-of-freedom system.



$$T = T_{tr} + T_{rot}$$

$$= \frac{1}{2} m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \frac{L_2^2}{4} \dot{\theta}_2^2 \right] + \frac{1}{2} \frac{m L_2^2}{12} \dot{\theta}_2^2$$

$$= \frac{1}{2} m \left[L_1^2 \dot{\theta}_1^2 + L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) + \frac{L_2^2}{3} \dot{\theta}_2^2 \right]$$

$$V = \int_{\mathbf{r}_C}^{\mathbf{r}_{C\text{ref}}} (-mg\mathbf{j}) \cdot d\mathbf{r}_C = -mg\mathbf{j} \cdot \mathbf{r}_C \Big|_{\mathbf{r}_C}^{\mathbf{r}_{C\text{ref}}}$$

$$= mg \left[L_1 (1 - \cos\theta_1) + \frac{L_2}{2} (1 - \cos\theta_2) \right] = mg \Delta h$$

$$\overline{\delta W}_{nc} = \mathbf{F} \cdot \delta \mathbf{r}_B = F\mathbf{i} \cdot \delta \left[(L_1 \sin\theta_1 + L_2 \sin\theta_2)\mathbf{i} - (L_1 \cos\theta_1 + L_2 \cos\theta_2)\mathbf{j} \right]$$

$$= F(L_1 \cos\theta_1 \delta\theta_1 + L_2 \cos\theta_2 \delta\theta_2) = \Theta_1 \delta\theta_1 + \Theta_2 \delta\theta_2 = Q_1 \delta q_1 + Q_2 \delta q_2$$

$$Q_1 = \Theta_1 = F L_1 \cos\theta_1, \; Q_2 = \Theta_2 = F L_2 \cos\theta_2$$

THE EXTENDED HAMILTON'S PRINCIPLE

$$\delta T = mL_1^2\dot\theta_1\delta\dot\theta_1 + \frac{mL_1L_2}{2}[\dot\theta_2\cos(\theta_2-\theta_1)\delta\dot\theta_1 + \dot\theta_1\cos(\theta_2-\theta_1)\delta\dot\theta_2$$

$$-\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1)\delta(\theta_2-\theta_1)] + \frac{mL_2^2}{3}\dot\theta_2\delta\dot\theta_2$$

$$= \frac{mL_1L_2}{2}\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1)\delta\theta_1 - \frac{mL_1L_2}{2}\dot\theta_1\dot\theta_2\sin(\theta_2-\theta_1)\delta\theta_2$$

$$+mL_1\left[L_1\dot\theta_1 + \frac{L_2}{2}\dot\theta_2\cos(\theta_2-\theta_1)\right]\delta\dot\theta_1 + mL_2\left[\frac{L_1}{2}\dot\theta_1\cos(\theta_1-\theta_1) + \frac{L_2}{3}\dot\theta_2\right]\delta\dot\theta_2$$

$$\delta V = mg\left(L_1\sin\theta_1\delta\theta_1 + \frac{L_2}{2}\sin\theta_2\delta\theta_2\right)$$

THE EXTENDED HAMILTON'S PRINCIPLE

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt = \int_{t_1}^{t_2} \Bigg\{ \Bigg[\frac{mL_1L_2}{2} \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - mgL_1 \sin\theta_1$$

$$+ FL_1 \cos\theta_1 \Bigg] \delta\theta_1 + \Bigg[-\frac{mL_1L_2}{2} \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - \frac{mgL_2}{2} \sin\theta_2 + FL_2 \cos\theta_2 \Bigg] \delta\theta_2$$

$$+ mL_1 \Bigg[L_1 \dot{\theta}_1 + \frac{L_2}{2} \dot{\theta}_2 \cos(\theta_2 - \theta_1) \Bigg] \delta\dot{\theta}_1 + mL_2 \Bigg[\frac{L_1}{2} \dot{\theta}_1 \cos(\theta_2 - \theta_1) + \frac{L_2}{3} \dot{\theta}_2 \Bigg] \delta\dot{\theta}_2 \Bigg\} dt = 0$$

Only the virtual displacements are arbitrary.

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THE EXTENDED HAMILTON'S PRINCIPLE

$$\int_{t_1}^{t_2} mL_1\left[L_1\dot{\theta}_1+\frac{L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right]\delta\dot{\theta}_1 dt = mL_1\left[L_1\dot{\theta}_1+\frac{L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right]\delta\theta_1\bigg|_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2} mL_1\frac{d}{dt}\left[L_1\dot{\theta}_1+\frac{L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)\right]\delta\theta_1 dt$$

$$=-\int_{t_1}^{t_2} mL_1\left[L_1\ddot{\theta}_1+\frac{L_2}{2}\ddot{\theta}_2\cos(\theta_2-\theta_1)-\frac{L_2}{2}\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)\right]\delta\theta_1\, dt$$

$$\int_{t_1}^{t_2} mL_2\left[\frac{L_1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)+\frac{L_2}{3}\dot{\theta}_2\right]\delta\dot{\theta}_2 dt = mL_2\left[\frac{L_1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)+\frac{L_2}{3}\dot{\theta}_2\right]\delta\theta_2\bigg|_{t_1}^{t_2}$$

$$-\int_{t_1}^{t_2} mL_2\frac{d}{dt}\left[\frac{L_1}{2}\dot{\theta}_1\cos(\theta_2-\theta_1)+\frac{L_2}{3}\dot{\theta}_2\right]\delta\theta_2 dt$$

$$=-\int_{t_1}^{t_2} mL_2\left[\frac{L_1}{2}\ddot{\theta}_1\cos(\theta_2-\theta_1)-\frac{L_1}{2}\dot{\theta}_1(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)+\frac{L_2}{3}\ddot{\theta}_2\right]\delta\theta_2 dt$$

THE EXTENDED HAMILTON'S PRINCIPLE

$\delta\theta_1 = \delta\theta_2 = 0$ at $t = t_1,\ t_2$.

$$\int_{t_1}^{t_2} \Bigg\{ -\left[mL_1^2\ddot{\theta}_1 + \frac{mL_1L_2}{2}\ddot{\theta}_2\cos(\theta_2-\theta_1) - \frac{mL_1L_2}{2}\dot{\theta}_2^2\sin(\theta_2-\theta_1) + mgL_1\sin\theta_1 - FL_1\cos\theta_1 \right]\delta\theta_1 - \left[\frac{mL_1L_2}{2}\ddot{\theta}_1\cos(\theta_2-\theta_1) + \frac{mL_2^2}{3}\ddot{\theta}_2 + \frac{mL_1L_2}{2}\dot{\theta}_1^2\sin(\theta_2-\theta_1) + \frac{mgL_2}{2}\sin\theta_2 - FL_2\cos\theta_2 \right]\delta\theta_2 \Bigg\} dt = 0$$

$$mL_1^2\ddot{\theta}_1 + \frac{mL_1L_2}{2}\left[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2^2\sin(\theta_2-\theta_1)\right] + mgL_1\sin\theta_1 = FL_1\cos\theta_1$$

$$\frac{mL_1L_2}{2}\left[\ddot{\theta}_1\cos(\theta_2-\theta_1) + \dot{\theta}_1^2\sin(\theta_2-\theta_1)\right] + \frac{mL_2^2}{3}\ddot{\theta}_2 + \frac{mgL_2}{2}\sin\theta_2 = FL_2\cos\theta_2$$

LAGRANGE'S EQUATIONS

For many problems the extended Hamilton's principle is not the most efficient method for deriving equations of motion:

- Involves routine operations that must be carried out every time the principle is applied,
  - The integrations by parts.

The extended Hamilton's principle is used to generate a more expeditious method for deriving equations of motion, *Lagrange's equations.*

LAGRANGE'S EQUATIONS

$$T = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$

$$\delta T = \sum_{k=1}^{n} \left(\frac{\partial T}{\partial q_k} \delta q_k + \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \right)$$

$$V = V(q_1, q_2, \ldots, q_n) \qquad \delta V = \sum_{k=1}^{n} \frac{\partial V}{\partial q_k} \delta q_k$$

$$\overline{\delta W}_{nc} = \sum_{k=1}^{n} Q_k \delta q_k$$

$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W}_{nc}) dt = \int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\left(\frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k} + Q_k \right) \delta q_k + \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k \right] dt = 0,$$

$$\delta q_k = 0, \ k = 1, 2, \ldots, n; \ t = t_1, t_2$$

LAGRANGE'S EQUATIONS

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_k}\delta\dot{q}_k dt = \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_k}\frac{d}{dt}\delta q_k\, dt = \frac{\partial T}{\partial \dot{q}_k}\delta q_k\bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial t}{\partial \dot{q}_k}\right)\delta q_k\, dt$$

$\delta q_k = 0,\ k = 1, 2, \ldots, n;\ t = t_1, t_2$ $\Longrightarrow$ $= -\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\delta q_k\, dt,\ k = 1, 2, \ldots, n$

$$\int_{t_1}^{t_2} \sum_{k=1}^{n}\left[\frac{\partial T}{\partial q_k} - \frac{\partial V}{\partial q_k} + Q_k - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right]\delta q_k\, dt = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k,\ k = 1, 2, \ldots, n$$

LAGRANGE'S EQUATIONS

Derive Lagrange's equations of motion for the system

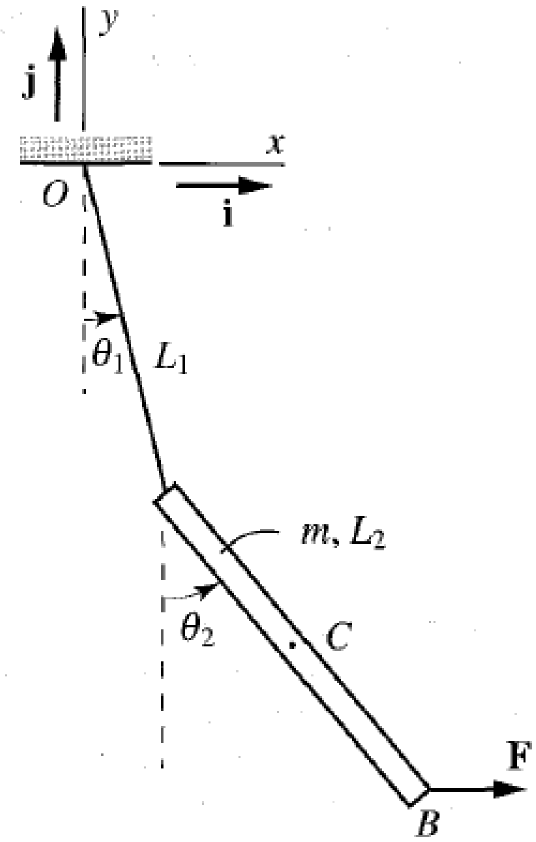
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_k}\right)-\frac{\partial T}{\partial \theta_k}+\frac{\partial V}{\partial \theta_k}=\Theta_k,\ k=1,2$$

$$V=mg\left[L_1(1-\cos\theta_1)+\frac{L_2}{2}(1-\cos\theta_2)\right]$$

$$\frac{\partial V}{\partial \theta_1}=mgL_1\sin\theta_1,\ \frac{\partial V}{\partial \theta_2}=\frac{mgL_2}{2}\sin\theta_2$$

$$\overline{\delta W}_{nc}=FL_1\cos\theta_1\delta\theta_1+FL_2\cos\theta_2\delta\theta_2$$

$$\Theta_1=FL_1\cos\theta_1,\ \Theta_2=FL_2\cos\theta_2$$



LAGRANGE'S EQUATIONS

$$T = \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{L_2^2}{3}\dot{\theta}_2^2\right]$$

$$\frac{\partial T}{\partial\dot{\theta}_1} = mL_1^2\dot{\theta}_1 + \frac{mL_1L_2}{2}\dot{\theta}_2\cos(\theta_2-\theta_1)$$

$$\frac{\partial T}{\partial\dot{\theta}_2} = \frac{mL_1L_2}{2}\dot{\theta}_1\cos(\theta_2-\theta_1) + \frac{mL_2^2}{3}\dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_1}\right) = mL_1^2\ddot{\theta}_1 + \frac{mL_1l_2}{2}\left[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)\right]$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}_2}\right) = \frac{mL_1L_2}{2}\left[\ddot{\theta}_1\cos(\theta_2-\theta_1) - \dot{\theta}_1(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)\right] + \frac{mL_2^2}{3}\ddot{\theta}_2$$

LAGRANGE'S EQUATIONS

$$T = \frac{1}{2}m\left[L_1^2\dot{\theta}_1^2 + L_1L_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2-\theta_1) + \frac{L_2^2}{3}\dot{\theta}_2^2\right]$$

$$\frac{\partial T}{\partial \theta_1} = \frac{mL_1L_2}{2}\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1), \quad \frac{\partial T}{\partial \theta_2} = -\frac{mL_1L_2}{2}\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)$$

$$mL_1^2\ddot{\theta}_1 + \frac{mL_1L_2}{2}[\ddot{\theta}_2\cos(\theta_2-\theta_1) - \dot{\theta}_2^2\sin(\theta_2-\theta_1)] + mgL_1\sin\theta_1 = FL_1\cos\theta_1$$

$$\frac{mL_1L_2}{2}[\ddot{\theta}_1\cos(\theta_2-\theta_1) + \dot{\theta}_1^2\sin(\theta_2-\theta_1)] + \frac{mL_2^2}{3}\ddot{\theta}_2 + \frac{mgL_2}{2}\sin\theta_2 = FL_2\cos\theta_2$$

Final word:

Lagrange's equations are more efficient, the extended Hamilton principle is more versatile.

In fact, it can produce results in cases in which Lagrange's equations cannot, most notably in the case of distributed-parameter systems.

Advanced Vibrations

Lecture Five:

MULTI-DEGREE-OF-FREEDOM SYSTEMS

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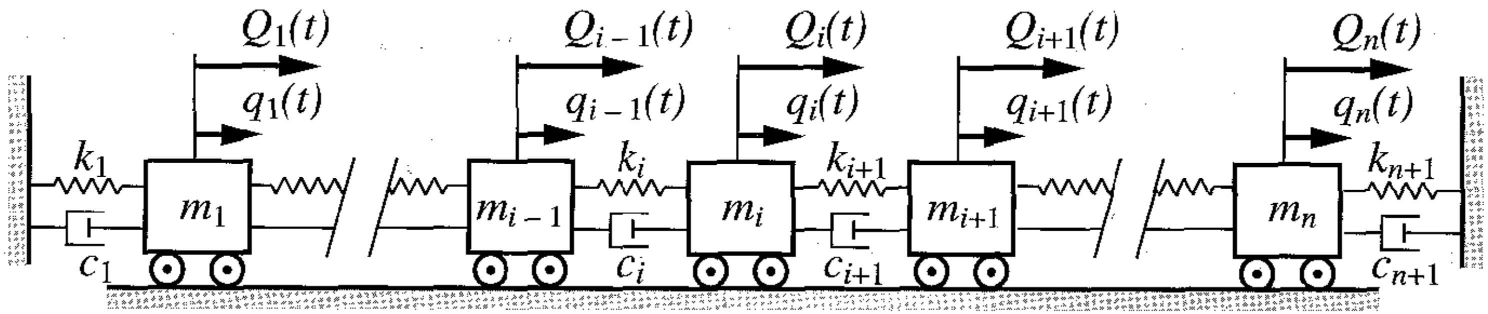
7.Multi-Degree-of-Freedom Systems

7.1 EQUATIONS OF MOTION FOR LINEAR SYSTEMS



$$\sum_{j=1}^{n}[m_{ij}\ddot{q}_j(t)+c_{ij}\dot{q}_j(t)+k_{ij}q_j(t)]=Q_i(t),\ i=1,2,\ldots,n$$

$$m_{ij}=m_{ji},\ c_{ij}=c_{ji},\ k_{ij}=k_{ji},$$

$$M=M^T,\qquad C=C^T,\qquad K=K^T$$

$$M\ddot{\mathbf{q}}(t)+C\dot{\mathbf{q}}(t)+K\mathbf{q}(t)=\mathbf{Q}(t)$$

7.2 FLEXIBILITY AND STIFFNESS INFLUENCE COEFFICIENTS

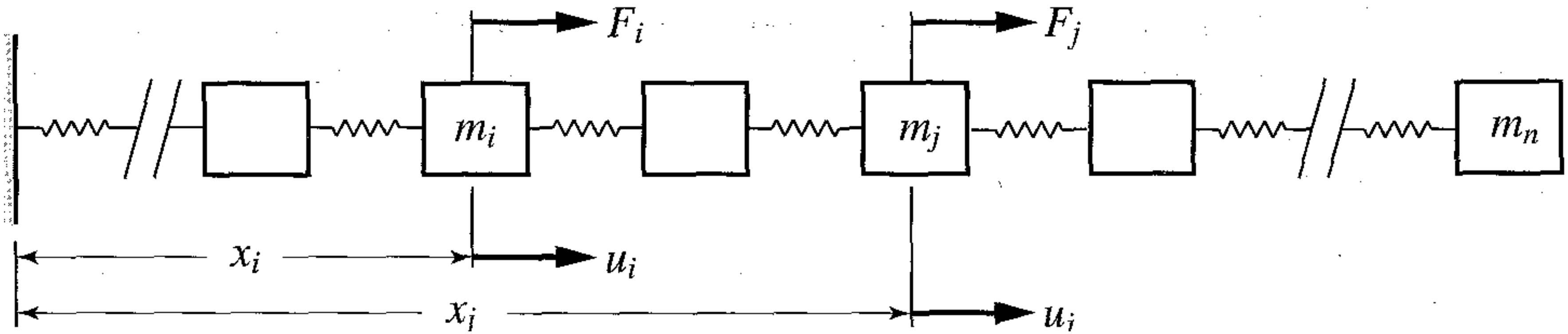
The stiffness coefficients can be obtained by other means, not necessarily involving the equations of motion.

- The stiffness coefficients are more properly known as *stiffness influence coefficients,* and can be derived by using its definition.

There is one more type of influence coefficients, namely, *flexibility influence coefficients.*

- They are intimately related to the stiffness influence coefficients.

7.2 FLEXIBILITY AND STIFFNESS INFLUENCE COEFFICIENTS



We *define the flexibility influence coefficient* $\boldsymbol{a_{ij}}$ *as the displacement of point* $\boldsymbol{x_i}$*, due to a unit force,* $\boldsymbol{F_j}$ *= 1.*

$$u_i = \sum_{j=1}^{n} a_{ij} F_j$$

7.2 FLEXIBILITY AND STIFFNESS INFLUENCE COEFFICIENTS

The stiffness influence coefficient k_{ij} *is the force required at* x_i *to produce a unit displacement at point* x_j*, and displacements at all other points are zero.*

- To obtain zero displacements at all points the forces must simply hold these points fixed.

$$F_i = \sum_{j=1}^{n} k_{ij} u_j$$

7.2 FLEXIBILITY AND STIFFNESS INFLUENCE COEFFICIENTS

$$[a_{ij}] = A, \; [k_{ij}] = K$$

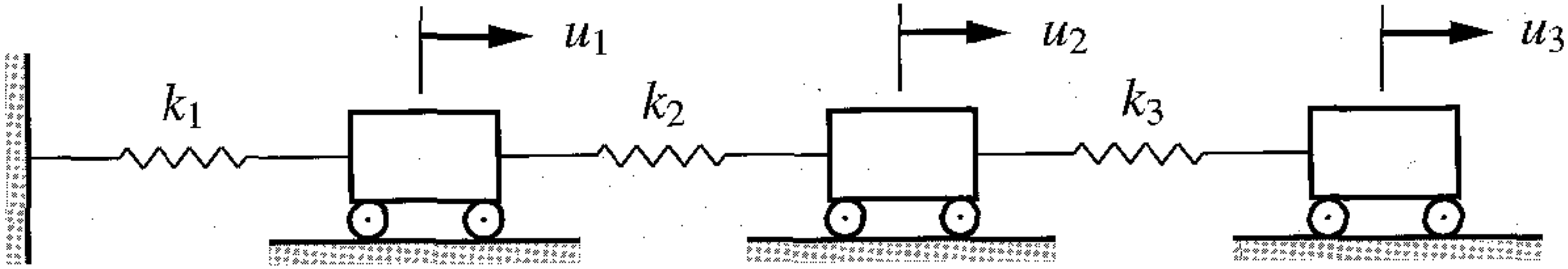
$$\mathbf{u} = A\mathbf{F}$$

$$\mathbf{F} = K\mathbf{u}$$

$$\mathbf{u} = A\mathbf{F} = AK\mathbf{u}$$

$$A = K^{-1}, \; K = A^{-1}$$

Example:



$$A = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{1}{k_1}+\frac{1}{k_2} & \frac{1}{k_1}+\frac{1}{k_2} \\ \frac{1}{k_1} & \frac{1}{k_1}+\frac{1}{k_2} & \frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3} \end{bmatrix}$$

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}$$

7.3 PROPERTIES OF THE STIFFNESS AND MASS COEFFICIENTS

The potential energy of a single linear spring:

$$V = \int_u^0 F_\zeta \, d\zeta = \int_u^0 (-k\zeta)\, d\zeta = \tfrac{1}{2} k u^2 = \tfrac{1}{2} F u$$

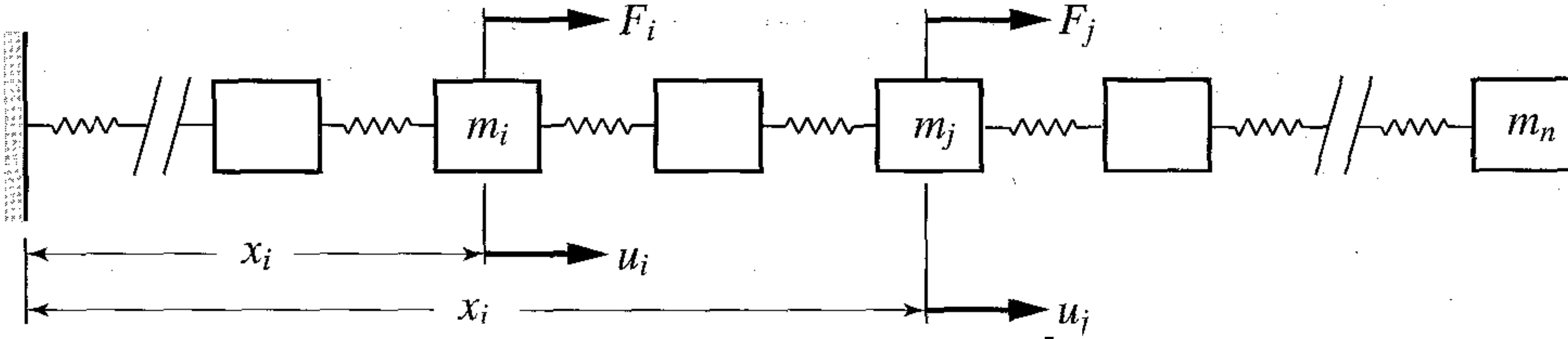
By analogy the elastic potential energy for a system is:

$$V = \sum_{i=1}^{n} V_i = \frac{1}{2} \sum_{i=1}^{n} F_i u_i$$

$$V = \frac{1}{2} \sum_{i=1}^{n} u_i \left(\sum_{j=1}^{n} k_{ij} u_j \right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij} u_i u_j$$

$$V = \frac{1}{2} \sum_{i=1}^{n} F_i \left(\sum_{j=1}^{n} a_{ij} F_j \right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} F_i F_j$$

Symmetry Property:



$$\tfrac{1}{2} F_i u_i' = \tfrac{1}{2} a_{ii} F_i^2$$

$$\tfrac{1}{2} F_i u_i' + F_i u_i'' + \tfrac{1}{2} F_j u_j'' = \tfrac{1}{2} a_{ii} F_i^2 + a_{ij} F_i F_j + \tfrac{1}{2} a_{jj} F_j^2$$

$$\tfrac{1}{2} F_j u_j'' = \tfrac{2}{2} a_{jj} F_j^2$$

$$\tfrac{1}{2} F_j u_j'' + F_j u_j' + \tfrac{1}{2} F_i u_i' = \tfrac{1}{2} a_{jj} F_j^2 + a_{ji} F_j F_i + \tfrac{1}{2} a_{ii} F_i^2$$

$$a_{ij} F_i F_j = a_{ji} F_j F_i$$

Maxwell's reciprocity theorem:

$$a_{ij} = a_{ji} \Longrightarrow$$

$$k_{ij} = k_{ji}, \quad i, j = 1, 2, \ldots, n$$

$$A = A^T, \quad K = K^T$$

$$V = \tfrac{1}{2}\mathbf{u}^T K \mathbf{u} \qquad\qquad V = \tfrac{1}{2}\mathbf{F}^T A \mathbf{F}$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{u}_i^2 \Longrightarrow T = \tfrac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}}$$

7.4 LAGRANGE'S EQUATIONS LINEARIZED ABOUT EQUILIBRIUM

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = Q_k, \quad k = 1, 2, \ldots, n$$

$$T = T(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \quad V = V(q_1, q_2, \ldots, q_n)$$

$$\overline{\delta W}_{nc} = \sum_{k=1}^{n} Q_k \delta q_k \qquad Q_{k\text{visc}} = -\frac{\partial \mathcal{F}}{\partial q_k}, \quad k = 1, 2, \ldots, n$$

Rayleigh's dissipation function

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} + \frac{\partial \mathcal{F}}{\partial \dot{q}_k} = Q_k, \quad k = 1, 2, \ldots, n$$

7.4 LAGRANGE'S EQUATIONS LINEARIZED ABOUT EQUILIBRIUM

$$q_k(t) = q_{ek} + \tilde{q}_k(t), \; k = 1, 2, \ldots, n$$

$$\dot{q}_k(t) = \dot{\tilde{q}}_k(t), \; k = 1, 2, \ldots, n$$

$$T \cong \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{q}=\mathbf{q}_e} \dot{\tilde{q}}_i \dot{\tilde{q}}_j = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}\dot{\tilde{q}}_i \dot{\tilde{q}}_j$$

$$m_{ij} = m_{ji} = \left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\mathbf{q}=\mathbf{q}_e}, \; i, j = 1, 2, \ldots, n$$

7.4 LAGRANGE'S EQUATIONS LINEARIZED ABOUT EQUILIBRIUM

$$V \cong V(\mathbf{q}_e) + \sum_{i=1}^{n} \frac{\partial V}{\partial q_i}\bigg|_{\mathbf{q}=\mathbf{q}_e} \tilde{q}_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{\mathbf{q}=\mathbf{q}_e} \tilde{q}_i \tilde{q}_j$$

$$= V(\mathbf{q}_e) + \sum_{i=1}^{n} \frac{\partial V}{\partial q_i}\bigg|_{\mathbf{q}=\mathbf{q}_e} \tilde{q}_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij} \tilde{q}_i \tilde{q}_j$$

$$k_{ij} = k_{ji} = \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{\mathbf{q}=\mathbf{q}_e}, \quad i, j = 1, 2, \ldots, n$$

7.4 LAGRANGE'S EQUATIONS LINEARIZED ABOUT EQUILIBRIUM

$$\mathcal{F} = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\dot{\tilde{q}}_i\dot{\tilde{q}}_j$$

$$\sum_{j=1}^{n}(m_{ij}\ddot{q}_j + c_{ij}\dot{q}_j + k_{ij}q_j) = Q_i, \quad i = 1, 2, \ldots, n$$

$$M\ddot{\mathbf{q}}(t) + C\dot{\mathbf{q}}(t) + K\mathbf{q} = \mathbf{Q}(t)$$

$$T = \tfrac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}}$$

$$\mathcal{F} = \tfrac{1}{2}\dot{\mathbf{q}}^T C \dot{\mathbf{q}}$$

$$\overline{\delta W}_{nc} = \mathbf{Q}^T \delta\mathbf{q}$$

$$V = \tfrac{1}{2}\mathbf{q}^T K \mathbf{q}$$

7.5 LINEAR TRANSFORMATIONS. COUPLING

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t)$$

$$\mathbf{q}(t) = U\eta(t)$$

$$\dot{\mathbf{q}}(t) = U\dot{\eta}(t), \ \ddot{\mathbf{q}}(t) = U\ddot{\eta}(t)$$

$$MU\ddot{\eta}(t) + KU\eta(t) = \mathbf{Q}(t)$$

$$M'\ddot{\eta}(t) + K'\eta(t) = \mathbf{N}(t)$$

$$M' = U^T MU = M'^T, \ K' = U^T KU = K'^T$$

$$\mathbf{N}(t) = U^T\mathbf{Q}(t)$$

Derivation of the matrices *M′* and K' in a more natural manner

$$T = \tfrac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}} \qquad V = \tfrac{1}{2}\mathbf{q}^T K \mathbf{q}$$

$$\mathbf{q}(t) = U\eta(t)$$

$$T = \tfrac{1}{2}\dot{\eta}^T(t) M' \dot{\eta}(t), \; V = \tfrac{1}{2}\eta^T(t) K' \eta(t)$$

$$M' = U^T M U = M'^T, \; K' = U^T K U = K'^T$$

$$M'_{jj}\ddot{\eta}_j(t) + K'_{jj}\eta_j(t) = N_j(t) \quad j = 1, 2, \ldots, n$$

7.Multi-Degree-of-Freedom Systems

Advanced Vibrations

Lecture 6
Multi-Degree-of-Freedom Systems (Ch7)

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7.Multi-Degree-of-Freedom Systems

7.6 UNDAMPED FREE VIBRATION. THE EIGENVALUE PROBLEM

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0}$$

$$\sum_{j=1}^{n} m_{ij}\ddot{q}_j(t) + \sum_{j=1}^{n} k_{ij}q_j(t) = 0, \quad i = 1, 2, \ldots, n$$

synchronous motion

$$q_j(t) = u_j f(t), \quad j = 1, 2, \ldots, n$$

$$\ddot{f}(t)\sum_{j=1}^{n} m_{ij}u_j + f(t)\sum_{j=1}^{n} k_{ij}u_j = 0, \quad i = 1, 2, \ldots, n$$

7.6 UNDAMPED FREE VIBRATION. THE EIGENVALUE PROBLEM

$$-\frac{\ddot{f}(t)}{f(t)} = \frac{\sum_{j=1}^{n} k_{ij} u_j}{\sum_{j=1}^{n} m_{ij} u_j}, \quad i = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} (k_{ij} - \lambda m_{ij}) u_j = 0, \quad i = 1, 2, \ldots, n$$

$$\ddot{f}(t) + \lambda f(t) = 0$$

$$f(t) = Ae^{st}$$

$$s^2 + \lambda = 0$$

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = \pm\sqrt{-\lambda}$$

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = \pm i\omega$$

$$f(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}$$

$$f(t) = C\cos(\omega t - \phi)$$

7.6 UNDAMPED FREE VIBRATION. THE EIGENVALUE PROBLEM

$$K\mathbf{u} = \omega^2 M\mathbf{u}$$

$$\Delta(\omega^2) = \det[K - \omega^2 M] = 0$$

characteristic polynomial

$$\omega_1 \le \omega_2 \le \cdots \le \omega_n$$

In general, all frequencies are distinct, except:

- In *degenerate* cases,
  - They cannot occur in one-dimensional structures;
  - They can occur in two-dimensional symmetric structures.

7.6 UNDAMPED FREE VIBRATION. THE EIGENVALUE PROBLEM

$$K\mathbf{u}_r = \omega_r^2 M\mathbf{u}_r, \; r = 1, 2, \ldots, n$$

The shape of the natural modes is unique but the amplitude is not.

A very convenient normalization scheme consists of setting:

$$\mathbf{u}_r^T M\mathbf{u}_r = 1, \; r = 1, 2, \ldots, n$$

$$\mathbf{u}_r^T K\mathbf{u}_r = \omega_r^2, \; r = 1, 2, \ldots, n$$

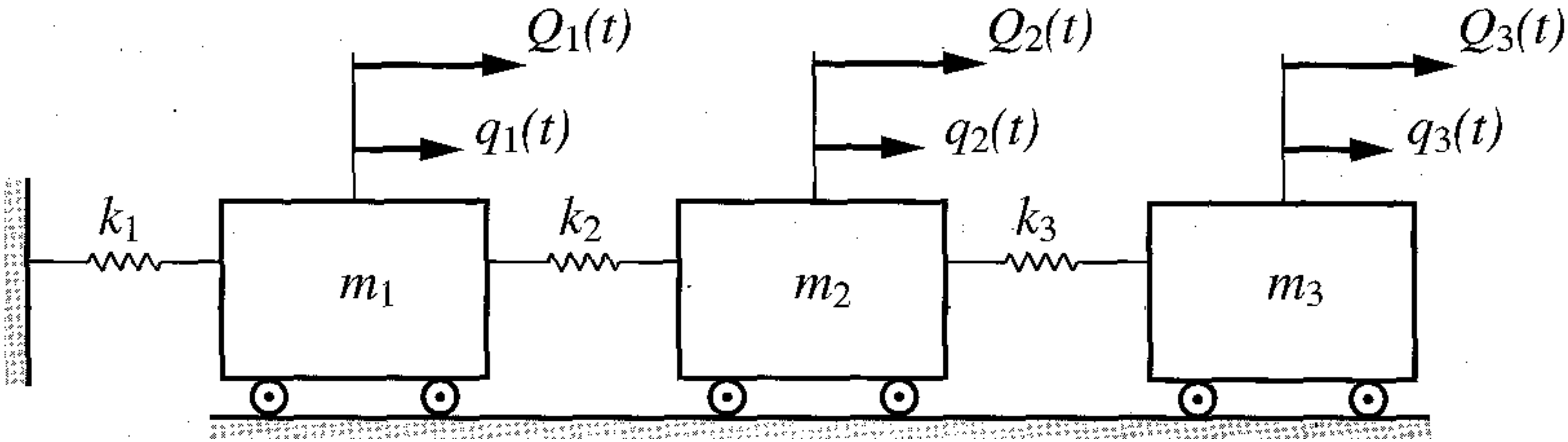
7.6 UNDAMPED FREE VIBRATION. THE EIGENVALUE PROBLEM

$$\mathbf{q}_r(t) = \mathbf{u}_r f_r(t), \; r = 1, 2, \ldots, n$$

$$f_r(t) = C_r \cos(\omega_r t - \phi_r), \; r = 1, 2, \ldots, n$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} \mathbf{q}_r(t) = \sum_{r=1}^{n} \mathbf{u}_r f_r(t) = U\mathbf{f}(t)$$

Free vibration for the initial excitations



$$\mathbf{q}(0) = q_0[1\ 2\ 3]^T,\ \dot{\mathbf{q}}(0) = \mathbf{0}.$$

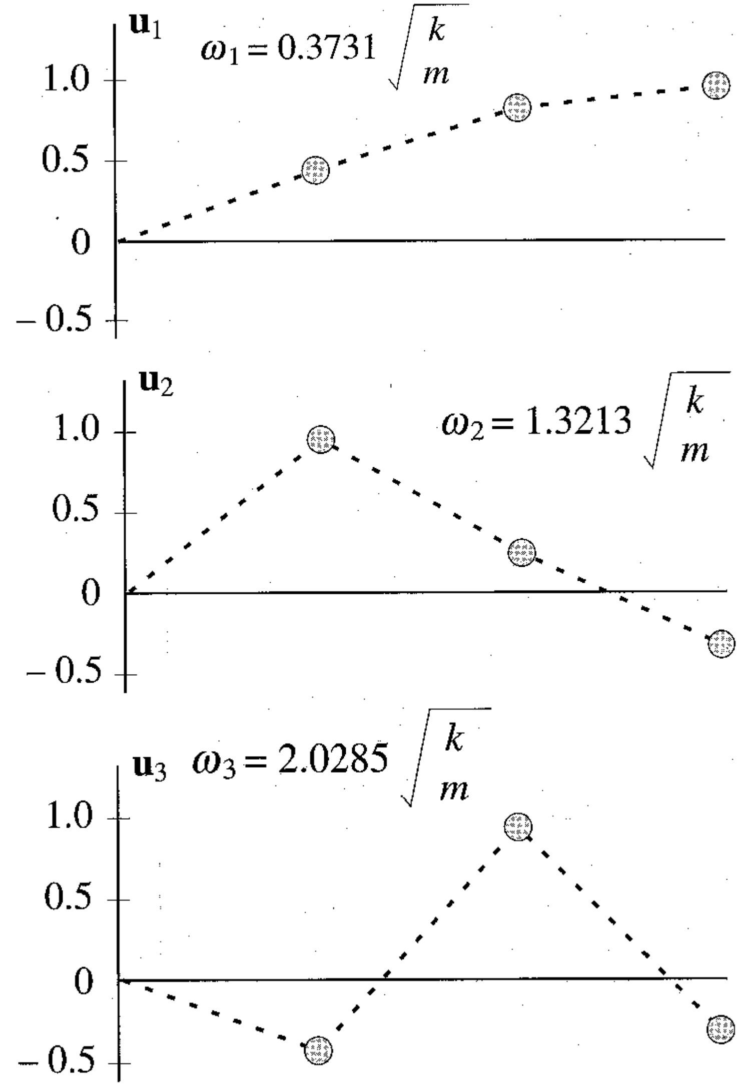
$$K = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \qquad M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Free vibration for the initial excitations

$$\Delta(\omega^2) = [K - \omega^2 M] =$$

$$\begin{bmatrix} 2k-\omega^2 m & -k & 0 \\ -k & 3k-\omega^2 m & -2k \\ 0 & -2k & 2(k-\omega^2 m) \end{bmatrix}$$

$$= -2m^3\left[\omega^6 - 6\frac{k}{m}\omega^4 + 8\left(\frac{k}{m}\right)^2\omega^2 - \left(\frac{k}{m}\right)^3\right] = 0$$



Free vibration for the initial excitations

$$\mathbf{q}(t) = \sum_{r=1}^{3} \mathbf{u}_r f_r(t) = \sum_{r=1}^{3} C_r \mathbf{u}_r \cos(\omega_r t - \phi_r)$$

$$= C_1 \begin{bmatrix} 0.4626 \\ 0.8608 \\ 1.0000 \end{bmatrix} \cos(0.3731\sqrt{\frac{k}{m}}t - \phi_1)$$

$$+ C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \cos\left(1.3213\sqrt{\frac{k}{m}}t - \phi_2\right)$$

$$+ C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \cos\left(2.0285\sqrt{\frac{k}{m}}t - \phi_3\right)$$

Free vibration for the initial excitations

$$\mathbf{q}(0) = C_1 \begin{bmatrix} 0.4626 \\ 0.8608 \\ 1.0000 \end{bmatrix} \cos\phi_1 + C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \cos\phi_2$$

$$+ C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \cos\phi_3 = q_0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$0.3731\sqrt{\frac{k}{m}}C_1 \begin{bmatrix} 0.4626 \\ 0.8608 \\ 1.0000 \end{bmatrix} \sin\phi_1 + 1.3213\sqrt{\frac{k}{m}}C_2 \begin{bmatrix} 1.0000 \\ 0.2541 \\ -0.3407 \end{bmatrix} \sin\phi_2$$

$$+2.0285\sqrt{\frac{k}{m}}C_3 \begin{bmatrix} -0.4728 \\ 1.0000 \\ -0.3210 \end{bmatrix} \sin\phi_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Free vibration for the initial excitations

$$\phi_1 = \phi_2 = \phi_3 = 0$$

$$C_1 = 2.7696q_0,\; C_2 = -0.4132q_0,\; C_3 = -0.2791q_0$$

$$\mathbf{x}(t) = q_0 \left\{ \begin{bmatrix} 1.2812 \\ 2.3841 \\ 2.7696 \end{bmatrix} \cos 0.3731\sqrt{\frac{k}{m}}t + \begin{bmatrix} -0.4132 \\ -0.1050 \\ 0.1408 \end{bmatrix} \cos 1.3213\sqrt{\frac{k}{m}}t \right.$$

$$\left. + \begin{bmatrix} 0.1320 \\ -0.2791 \\ 0.0896 \end{bmatrix} \cos 2.0285\sqrt{\frac{k}{m}}t \right\}$$

7.7 ORTHOGONALITY OF MODAL VECTORS

$$K\mathbf{u}_r = \omega_r^2 M\mathbf{u}_r, \quad K\mathbf{u}_s = \omega_s^2 M\mathbf{u}_s$$

$$\mathbf{u}_s^T K\mathbf{u}_r = \omega_r^2 \mathbf{u}_s^T M\mathbf{u}_r$$

$$\mathbf{u}_r^T K\mathbf{u}_s = \omega_s^2 \mathbf{u}_r^T M\mathbf{u}_s$$

$$(\omega_r^2 - \omega_s^2)\mathbf{u}_s^T M\mathbf{u}_r = 0$$

$$\mathbf{u}_s^T M\mathbf{u}_r = 0, \mathbf{u}_s^T K\mathbf{u}_r = 0, \; r \neq s$$

7.7 ORTHOGONALITY OF MODAL VECTORS

$$\mathbf{u}_r^T M \mathbf{u}_s = \delta_{rs}, \ \mathbf{u}_r^T K \mathbf{u}_s = \omega_r^2 \delta_{rs}, \ r, s = 1, 2, \ldots, n$$

$$KU = MU\Omega$$

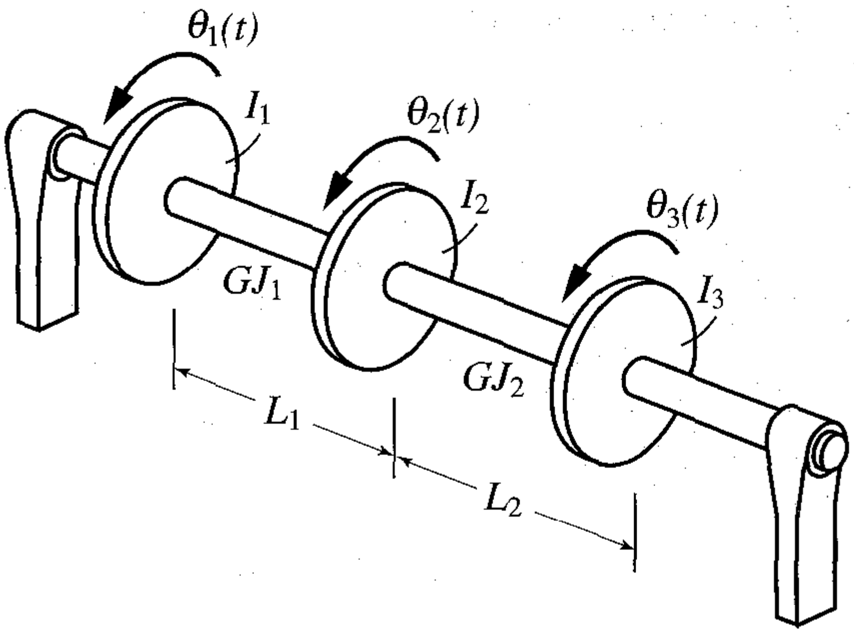
$$U^T MU = I, \ U^T KU = \Omega$$

7.8 SYSTEMS ADMITTING RIGID-BODY MOTIONS

$$T = \tfrac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2) = \tfrac{1}{2}\dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}}$$

$$V = \tfrac{1}{2}[k_1(\theta_2 - \theta_1)^2 + k_2(\theta_3 - \theta_2)^2] = \tfrac{1}{2}\boldsymbol{\theta}^T K \boldsymbol{\theta}$$

$$M = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$



$$K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

7.8 SYSTEMS ADMITTING RIGID-BODY MOTIONS

$$\boldsymbol{\Theta} = \boldsymbol{\Theta}_0 = \Theta_0 [1 \quad 1 \quad 1]^T$$

$$K\boldsymbol{\Theta}_0 = \Theta_0 \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

$$\boldsymbol{\Theta}_0^T M \boldsymbol{\Theta} = \Theta_0 (I_1 \Theta_1 + I_2 \Theta_2 + I_3 \Theta_3) = 0$$

$$\boxed{I_1 \dot{\theta}_1(t) + I_2 \dot{\theta}_2(t) + I_3 \dot{\theta}_3(t) = 0}$$

The orthogonality of the rigid-body mode to the elastic modes is equivalent to the preservation of zero angular momentum in pure elastic motion.

$$\boxed{\theta_3 = -\frac{I_1}{I_3}\theta_1 - \frac{I_2}{I_3}\theta_2}$$

7.8 SYSTEMS ADMITTING RIGID-BODY MOTIONS

$$\boldsymbol{\theta} = [\theta_1 \quad \theta_2 \quad \theta_3]^T \qquad \boldsymbol{\theta}' = [\theta_1 \quad \theta_2]^T$$

$$\boldsymbol{\theta} = C\boldsymbol{\theta}' \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{-I_1}{I_3} & \dfrac{-I_2}{I_3} \end{bmatrix}$$

$$T = \tfrac{1}{2}\dot{\boldsymbol{\theta}}^T M \dot{\boldsymbol{\theta}} = \tfrac{1}{2}\dot{\boldsymbol{\theta}}'^T C^T M C \dot{\boldsymbol{\theta}}' = \tfrac{1}{2}\dot{\boldsymbol{\theta}}'^T M' \dot{\boldsymbol{\theta}}'$$

$$V = \tfrac{1}{2}\boldsymbol{\theta}^T K \boldsymbol{\theta} = \tfrac{1}{2}\boldsymbol{\theta}'^T C^T K C \boldsymbol{\theta}' = \tfrac{1}{2}\boldsymbol{\theta}'^T K' \boldsymbol{\theta}'$$

7.8 SYSTEMS ADMITTING RIGID-BODY MOTIONS

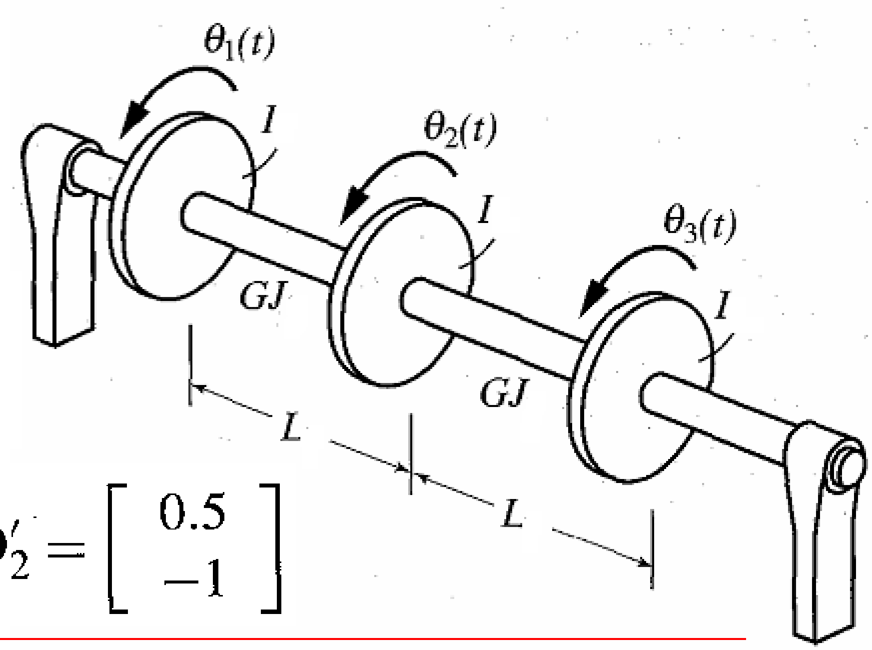
$$M' = C^T M C = \frac{1}{I_3}\begin{bmatrix} I_1(I_1+I_3) & I_1 I_2 \\ I_1 I_2 & I_2(I_2+I_3) \end{bmatrix}$$

$$K' = C^T K C =$$

$$\frac{1}{I_3^2}\begin{bmatrix} k_1 I_3^2 + k_2 I_1^2 & -k_1 I_3^2 + k_2 I_1(I_2+I_3) \\ -k_1 I_3^2 + k_2 I_1(I_2+I_3) & k_1 I_3^2 + k_2(I_2+I_3)^2 \end{bmatrix}$$

$$K'\boldsymbol{\Theta}' = \omega^2 M' \boldsymbol{\Theta}'$$

7.8 SYSTEMS ADMITTING RIGID-BODY MOTIONS



$$M' = \frac{1}{I}\begin{bmatrix} 2I^2 & I^2 \\ I^2 & 2I^2 \end{bmatrix} = I\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$K' = \frac{1}{I^2}\begin{bmatrix} 2kI^2 & kI^2 \\ kI^2 & 5kI^2 \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

$$\omega_1 = \sqrt{\frac{k}{I}},\ \mathbf{\Theta}_1' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \omega_2 = \sqrt{\frac{3k}{I}},\ \mathbf{\Theta}_2' = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

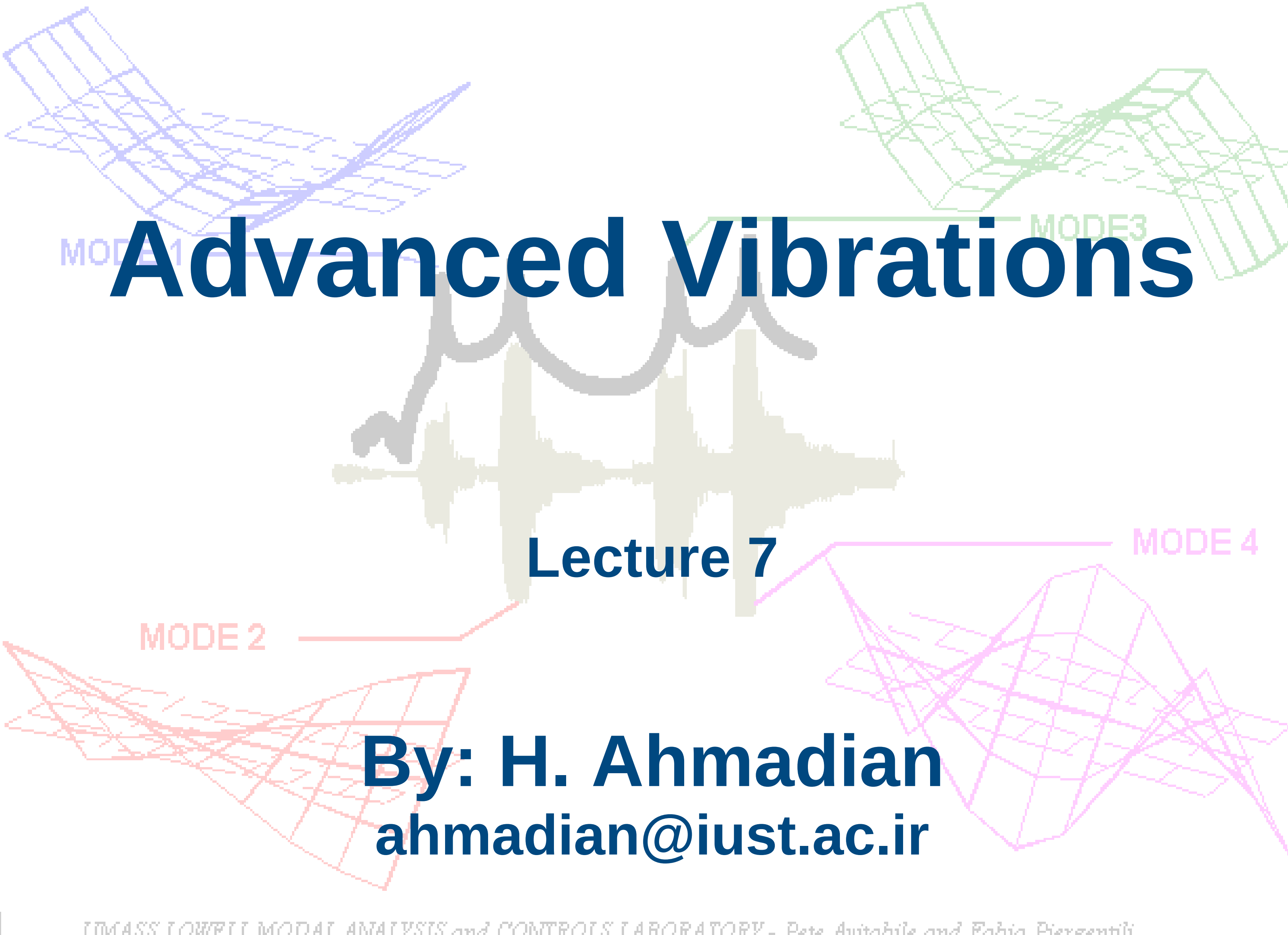
$$\mathbf{\Theta}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{\Theta}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{\Theta}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$$

7.Multi-Degree-of-Freedom Systems

Advanced Vibrations



Lecture 7

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7.Multi-Degree-of-Freedom Systems

7.1 Equations of Motion for Linear Systems

7.2 Flexibility and Stiffness Influence Coefficients

7.3 Properties of the Stiffness and Mass Coefficients

7.4 Lagrange's Equations Linearized about Equilibrium

7.5 Linear Transformations : Coupling

7.6 Undamped Free Vibration :The Eigenvalue Problem

7.7 Orthogonality of Modal Vectors

7.8 Systems Admitting Rigid-Body Motions

7.9 Decomposition of the Response in Terms of Modal Vectors

7.10 Response to Initial Excitations by Modal Analysis

7.11 Eigenvalue Problem in Terms of a Single Symmetric Matrix

7.12 Geometric Interpretation of the Eigenvalue Problem

7.13 Rayleigh's Quotient and Its Properties

7.14 Response to Harmonic External Excitations

7.15 Response to External Excitations by Modal Analysis

- **7.15.1** Undamped systems
- **7.15.2** Systems with proportional damping

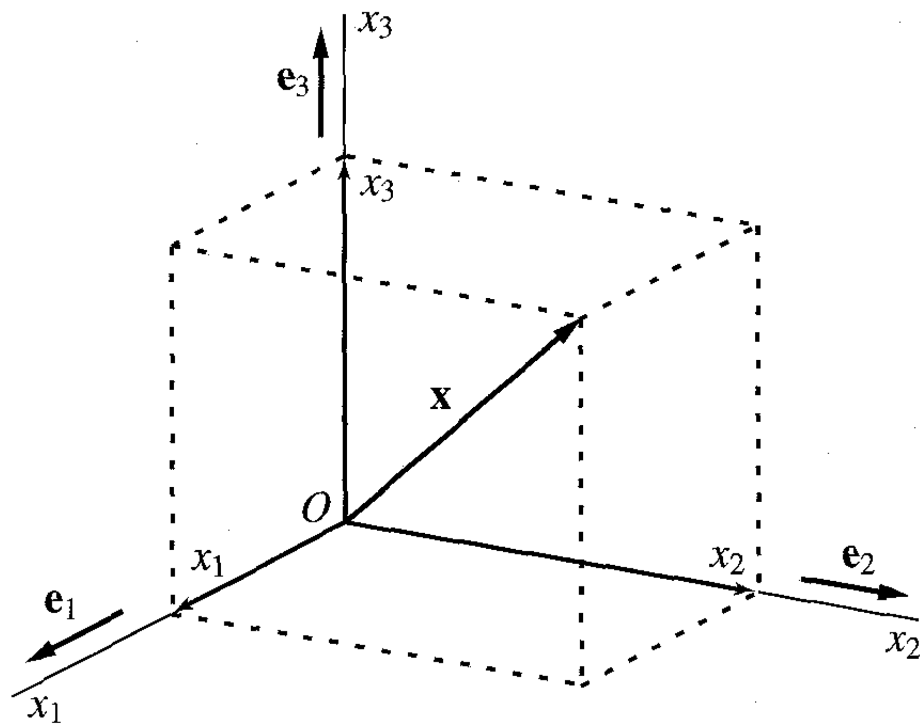
7.16 Systems with Arbitrary Viscous Damping

7.17 Discrete-Time Systems

7.9 Decomposition of the Response in Terms of Modal Vectors

$$\mathbf{x} = [x_1 \;\; x_2 \;\; \dots \;\; x_n]^T = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

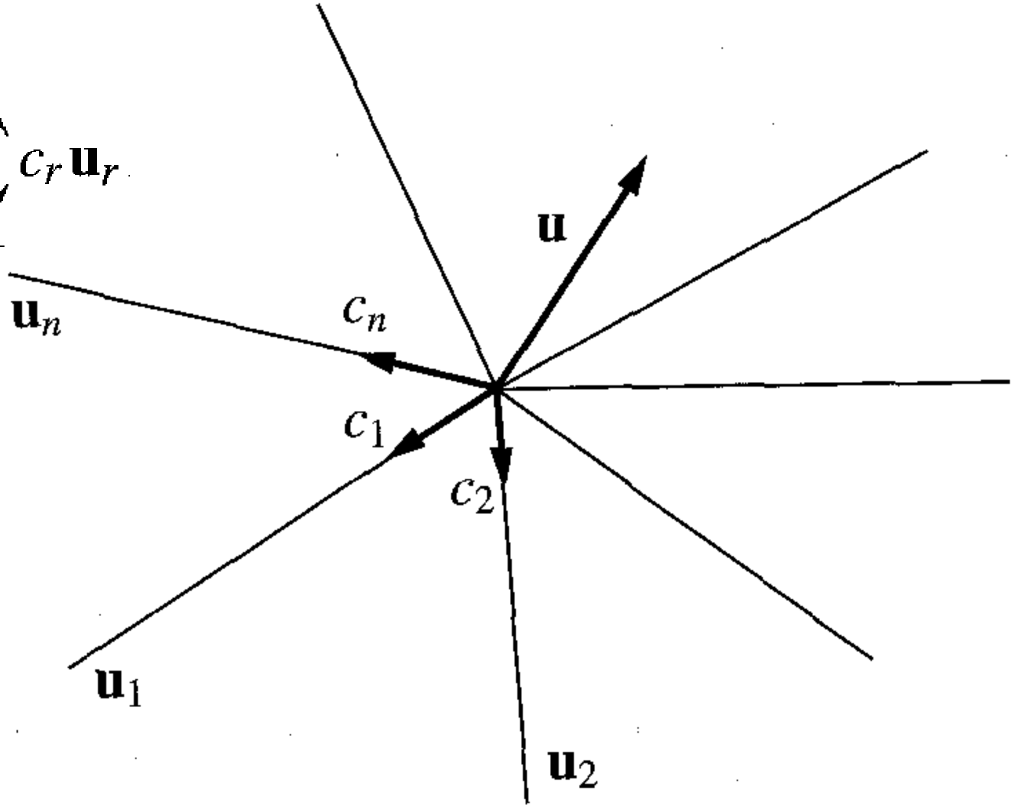
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \; \dots, \; \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$



7.9 Decomposition of the Response in Terms of Modal Vectors

The modal vectors are orthonormal with respect to the mass matrix *M*,

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = \sum_{r=1}^{n} c_r\mathbf{u}_r$$



$$c_r = \mathbf{u}_r^T M\mathbf{u},\ r = 1, 2, \ldots, n$$

$$\mathbf{u} = U\mathbf{c}$$

$$\mathbf{c} = U^T M\mathbf{u},\ \Omega\mathbf{c} = U^T K\mathbf{u}$$

7.10 Response to Initial Excitations by Modal Analysis

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0}$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} \eta_r(t)\mathbf{u}_r$$

$$\eta_r(t) = \mathbf{u}_r^T M\mathbf{q}(t),\ \omega_r^2\eta_r(t) = \mathbf{u}_r^T K\mathbf{q}(t),\ r = 1, 2, \ldots, n$$

7.10 Response to Initial Excitations by Modal Analysis

$$\mathbf{q}(t) = \sum_{r=1}^{n} \eta_r(t)\mathbf{u}_r \quad \Rightarrow \quad \mathbf{q}(t) = U\boldsymbol{\eta}(t)$$

Modal Coordinates

$$\ddot{\boldsymbol{\eta}}(t) + \Omega\boldsymbol{\eta}(t) = \mathbf{0}$$

$$\ddot{\eta}_r(t) + \omega_r^2\eta_r(t) = 0, \quad r = 1, 2, \ldots, n$$

$$\eta_r(t) = C_r \cos(\omega_r t - \phi_r)$$

$$= \eta_r(0)\cos\omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t,$$

7.10 Response to Initial Excitations by Modal Analysis

$$\eta_r(t) = \mathbf{u}_r^T M \mathbf{q}(0) \cos \omega_r t + \frac{1}{\omega_r} \mathbf{u}_r^T M \dot{\mathbf{q}}(0) \sin \omega_r t,$$

$$r = 1, 2, \ldots, n$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} [\mathbf{u}_r^T M \mathbf{q}(0) \cos \omega_r t + \frac{1}{\omega_r} \mathbf{u}_r^T M \dot{\mathbf{q}}(0) \sin \omega_r t] \mathbf{u}_r$$

7.10 Response to Initial Excitations by Modal Analysis

We wish to demonstrate that each of the natural modes can be excited independently of the other;

$$\mathbf{q}(0) = \alpha \mathbf{u}_s, \quad \dot{\mathbf{q}}(0) = \mathbf{0}$$

$$\mathbf{q}(t) = \sum^{n} [\mathbf{u}_r^T M \mathbf{q}(0) \cos \omega_r t + \frac{1}{\omega_r} \mathbf{u}_r^T M \dot{\mathbf{q}}(0) \sin \omega_r t] \mathbf{u}_r$$

$$\mathbf{q}(t) = \alpha \sum_{r=1}^{n} [\mathbf{u}_r^T M \mathbf{u}_s \cos \omega_r t] \mathbf{u}_r$$

$$= \alpha \sum_{r=1}^{n} \delta_{rs} \mathbf{u}_r \cos \omega_r t = \alpha \mathbf{u}_s \cos \omega_s t$$

7.11 Eigenvalue Problem in Terms of a Single Symmetric Matrix

$$K\mathbf{u} = \omega^2 M\mathbf{u}$$

$$M = LL^T \qquad K\mathbf{u} = \omega^2 LL^T\mathbf{u}$$

$$L^T\mathbf{u} = \mathbf{v}$$

$$A\mathbf{v} = \lambda\mathbf{v},\ \lambda = \omega^2$$

$$A = L^{-1}K(L^{-1})^T = A^T$$

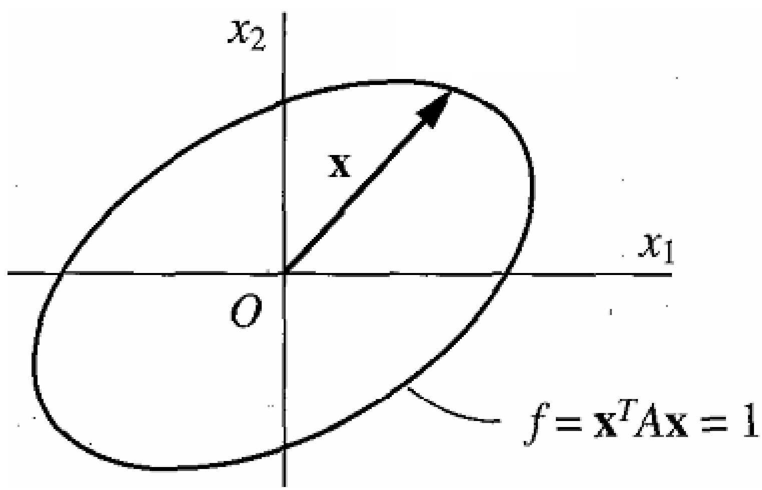
$$\mathbf{v}_s^T\mathbf{v}_r = \delta_{rs},\ \mathbf{v}_s^T A\mathbf{v}_r = \lambda_r\delta_{rs},\ r,s = 1,\ 2,\ldots,n$$

7.12 Geometric Interpretation of the Eigenvalue Problem

$$f = \mathbf{x}^T A\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j$$

$$n = 2$$

$$f = \mathbf{x}^T A\mathbf{x} = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2 = 1$$



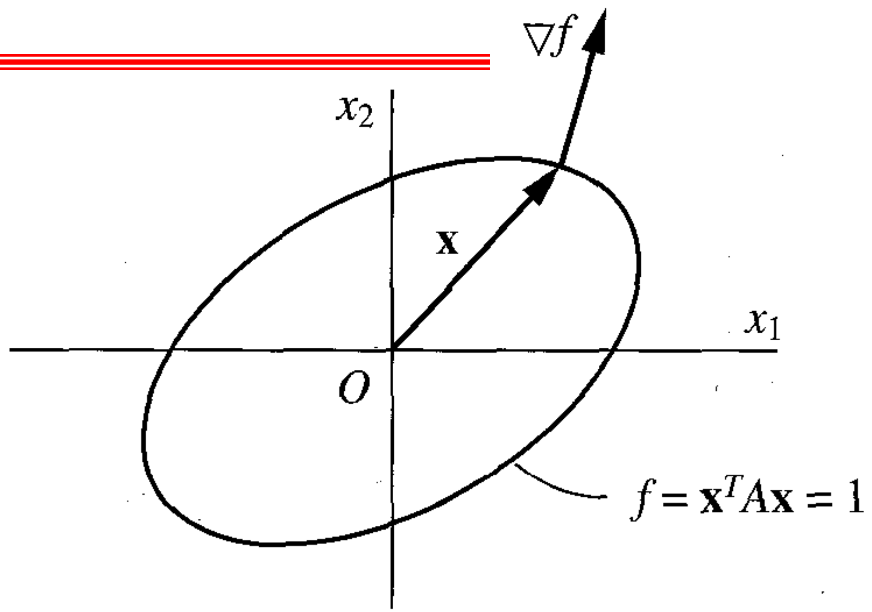
7.12 Geometric Interpretation of the Eigenvalue Problem

$$\nabla f = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = 2\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{12}x_1 + a_{22}x_2 \end{bmatrix}$$

$$= 2\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2A\mathbf{x}$$

$$\nabla f = 2\lambda\mathbf{x}$$

$$A\mathbf{x} = \lambda\mathbf{x}$$



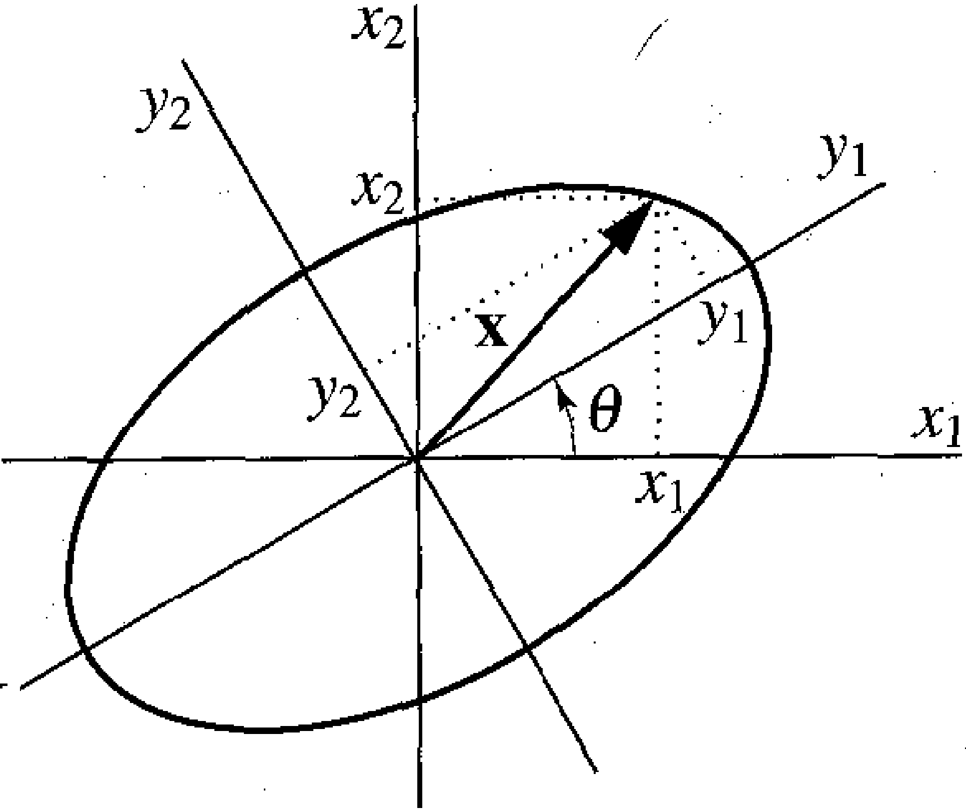
7.12 Geometric Interpretation of the Eigenvalue Problem

Solving the eigenvalue problem by finding the principle axes of the ellipse.

$$\mathbf{x} = R\mathbf{y}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R^T R = R R^T = I$$



7.12 Geometric Interpretation of the Eigenvalue Problem

$$f = \mathbf{x}^T A\mathbf{x} = \mathbf{y}^T R^T A R\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 1$$

Transforming to canonical form implies elimination of cross products:

$$D = R^T A R = \mathrm{diag}[d_1 \;\; d_2]$$

$$D = \Lambda, \;\; R = V$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

7.12 Geometric Interpretation of the Eigenvalue Problem

$$\lambda_1 = a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta$$

$$\lambda_2 = a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta$$

$$0 = -(a_{11} - a_{22})\sin\theta\cos\theta + a_{12}(\cos^2\theta - \sin^2\theta)$$

$$\sin 2\theta = 2\sin\theta\cos\theta \text{ and } \cos 2\theta = \cos^2\theta - \sin^2\theta,$$

$$\boxed{\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}}}$$

7.12 Geometric Interpretation of the Eigenvalue Problem

Obtaining the angle, one may calculate the eigenvalues and eigenvectors:

$$\lambda_1 = a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta$$

$$\lambda_2 = a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta$$

$$\mathbf{v}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

7.12 Geometric Interpretation of the Eigenvalue Problem

Example: Solving the eigenvalue problem by finding the principal axes of the corresponding ellipse.

$$M = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad K = \frac{T}{L}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix}$$

7.12 Geometric Interpretation of the Eigenvalue Problem

$$A = \begin{bmatrix} 2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 3/2 \end{bmatrix} \qquad \lambda = \omega^2 mL/T$$

$$\tan 2\theta = \frac{2a_{12}}{a_{11} - a_{22}} \quad b = a_{12} = -\frac{1}{\sqrt{2}} \quad c = \frac{1}{2}(a_{11} - a_{22}) = \frac{1}{4}$$

$$\cos\theta = \left[\frac{1}{2} + \frac{c}{2(b^2 + c^2)^{1/2}}\right]^{1/2} = 0.816497$$

$$\sin\theta = \frac{b}{2(b^2 + c^2)^{1/2}\cos\theta} = -0.577350$$

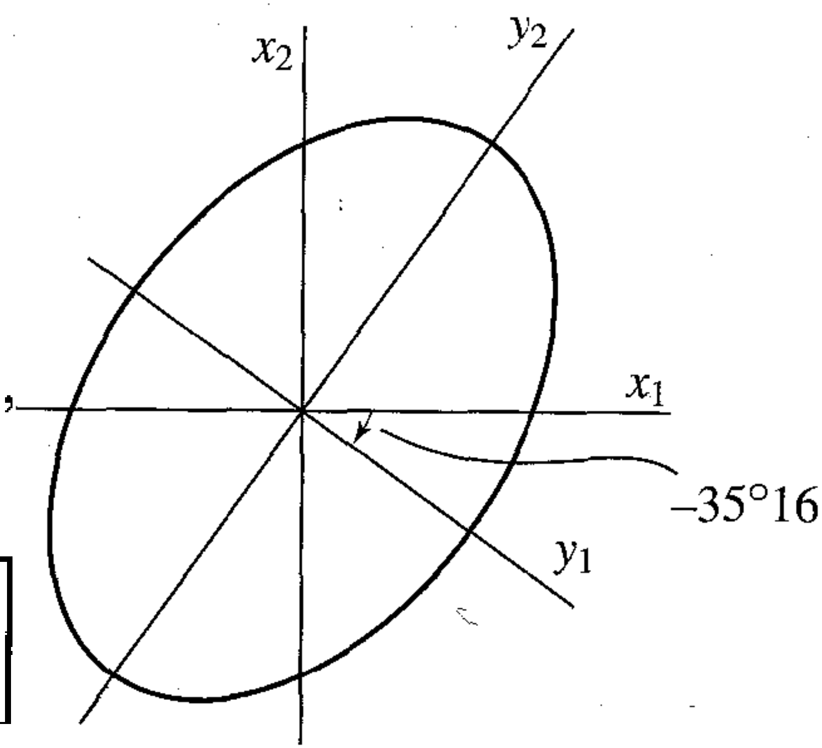
7.12 Geometric Interpretation of the Eigenvalue Problem

$$\lambda_1 = a_{11}\cos^2\theta + 2a_{12}\sin\theta\cos\theta + a_{22}\sin^2\theta$$
$$= 2.5$$

$$\lambda_2 = a_{11}\sin^2\theta - 2a_{12}\sin\theta\cos\theta + a_{22}\cos^2\theta$$
$$= 1$$

$$\mathbf{v}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} 0.816497 \\ -0.577350 \end{bmatrix},$$

$$\mathbf{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} 0.577350 \\ 0.816497 \end{bmatrix}$$



7.Multi-Degree-of-Freedom Systems

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Advanced Vibrations

Lecture 8

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Iran University of Science and Technology

7.Multi-Degree-of-Freedom Systems

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

$$K\mathbf{u}_r = \lambda_r M\mathbf{u}_r,\ \lambda_r = \omega_r^2,\ r = 1, 2, \ldots, n$$

$$\lambda_r = \omega_r^2 = \frac{\mathbf{u}_r^T K\mathbf{u}_r}{\mathbf{u}_r^T M\mathbf{u}_r},\ r = 1, 2, \ldots, n$$

$$R(\mathbf{u}) = \lambda = \omega^2 = \frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}}$$

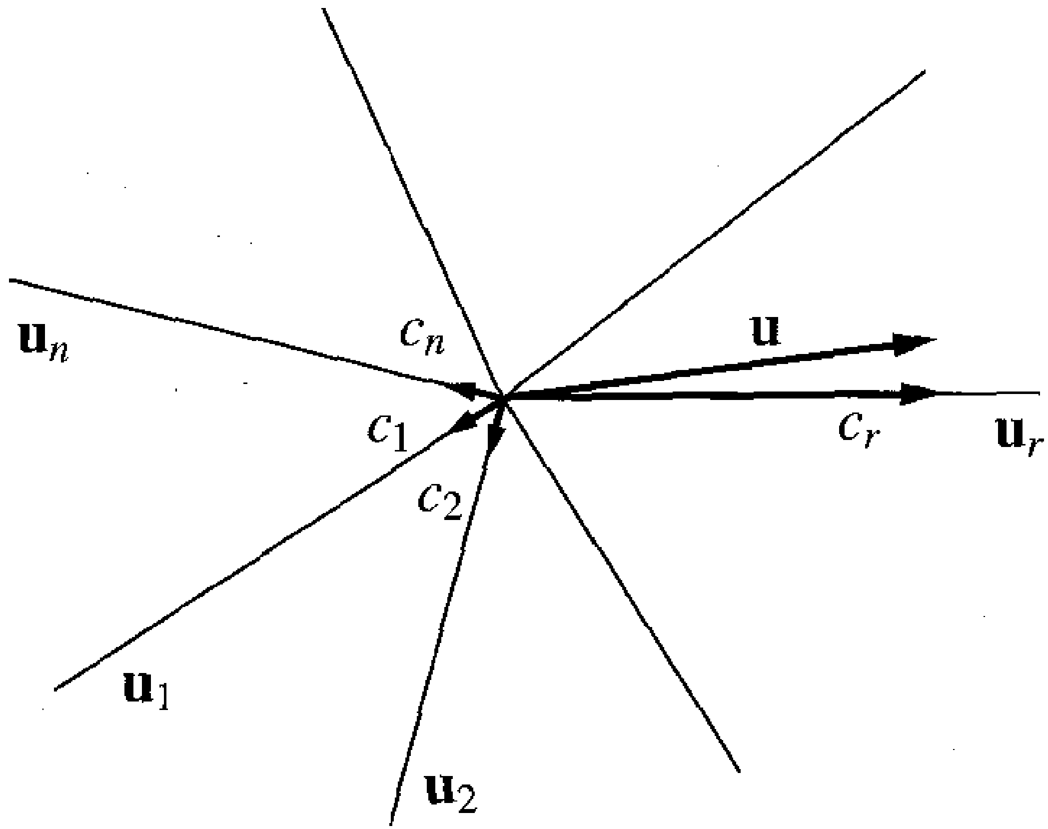
Rayleigh's quotient

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

$$\mathbf{u} = \sum_{r=1}^{n} c_r \mathbf{u}_r = U\mathbf{c}$$

$$R = \frac{\mathbf{u}^T K \mathbf{u}}{\mathbf{u}^T M \mathbf{u}}$$

$$= \frac{\mathbf{c}^T U^T K U \mathbf{c}}{\mathbf{c}^T U^T M U \mathbf{c}}$$



7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

$$R = \frac{\mathbf{u}^T K \mathbf{u}}{\mathbf{u}^T M \mathbf{u}} = \frac{\mathbf{c}^T U^T K U \mathbf{c}}{\mathbf{c}^T U^T M U \mathbf{c}} = \frac{\mathbf{c}^T \Lambda \mathbf{c}}{\mathbf{c}^T \mathbf{c}} = \frac{\sum_{i=1}^{n} \lambda_i c_i^2}{\sum_{i=1}^{n} c_i^2}$$

$$U^T M U = I, \; U^T K U = \Lambda$$

$$c_i = \epsilon_i c_r, \; i = 1, 2, \ldots, n; \; i \neq r$$

$$R = \frac{\lambda_r + \sum_{\substack{i=1 \\ i \neq r}}^{n} \lambda_i \epsilon_i^2}{1 + \sum_{\substack{i=1 \\ i \neq r}}^{n} \epsilon_i^2}$$

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

$$R = \frac{\lambda_r + \sum\limits_{\substack{i=1 \\ i \neq r}}^{n} \lambda_i \epsilon_i^2}{1 + \sum\limits_{\substack{i=1 \\ i \neq r}}^{n} \epsilon_i^2} \cong \left(\lambda_r + \sum_{\substack{i=1 \\ i \neq r}}^{n} \lambda_i \epsilon_i^2 \right)\left(1 - \sum_{\substack{i=1 \\ i \neq r}}^{n} \epsilon_i^2 \right)$$

$$\cong \lambda_r + \sum_{i=1}^{n} (\lambda_i - \lambda_r)\epsilon_i^2$$

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

Of special interest in vibrations is the fundamental frequency.

$$R \cong \lambda_1 + \sum_{i=2}^{n} (\lambda_i - \lambda_1)\epsilon_i^2 \Longrightarrow R \geq \lambda_1$$

Rayleigh's quotient is an upper bound for the lowest eigenvalue.

$$\lambda_1 = \min_{\mathbf{u}} R(\mathbf{u}) = \min_{\mathbf{u}} \frac{\mathbf{u}^T K \mathbf{u}}{\mathbf{u}^T M \mathbf{u}}$$

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

Example:

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad K = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\mathbf{F} = c[m_1 \;\; m_2 \;\; m_3]^T = [1 \;\; 1 \;\; 2]^T$$

Simulates gravity loading

$$\mathbf{u} = \frac{1}{k}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{k}\begin{bmatrix} 4 \\ 7 \\ 8 \end{bmatrix}$$

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

$$R = \omega^2 = \frac{\mathbf{u}^T K \mathbf{u}}{\mathbf{u}^T M \mathbf{u}} = \frac{27k}{193m} = 0.1399\frac{k}{m}$$

$$\omega = 0.3740\sqrt{\frac{k}{m}} \qquad \omega_1 = 0.3731\sqrt{\frac{k}{m}}$$

Exact solution

$$\frac{\omega - \omega_1}{\omega_1} = \frac{0.3740 - 0.3731}{0.3731} = 0.002412 = 0.2412\%$$

7.13 RAYLEIGH'S QUOTIENT AND ITS PROPERTIES

$$\mathbf{u} = \begin{bmatrix} 0.3522 \\ 0.6163 \\ 0.7044 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 0.3309 \\ 0.6155 \\ 0.7152 \end{bmatrix}$$

$$\frac{\|\mathbf{u} - \mathbf{u}_1\|}{\|\mathbf{u}_1\|} = \sqrt{(\mathbf{u} - \mathbf{u}_1)^T (\mathbf{u} - \mathbf{u}_1)} = 0.0239 = 2.39\%$$

7.14 RESPONSE TO HARMONIC EXTERNAL EXCITATIONS

$$M\ddot{\mathbf{q}}(t) + C\dot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t)$$

$$\mathbf{Q}(t) = \mathbf{Q}_0 e^{i\alpha t} \qquad \mathbf{q}(t) = \mathbf{q}_0 e^{i\alpha t}$$

$$(-\alpha^2 M + i\alpha C + K)\mathbf{q}_0 e^{i\alpha t} = \mathbf{Q}_0 e^{i\alpha t}$$

$$Z(i\alpha)\mathbf{q}_0 = \mathbf{Q}_0$$

$$Z(i\alpha) = -\alpha^2 M + i\alpha C + K$$

$$\mathbf{q}_0 = Z^{-1}(i\alpha)\mathbf{Q}_0$$

7.14 RESPONSE TO HARMONIC EXTERNAL EXCITATIONS

$$\mathbf{q}_0 = Z^{-1}(i\alpha)\mathbf{Q}_0$$

$$Z^{-1}(i\alpha) = G(i\alpha)$$

$$\mathbf{q}(t) = G(i\alpha)\mathbf{Q}_0 e^{i\alpha t}$$

This approach is feasible only for systems with a small number of degrees of freedom.

For large systems, it becomes necessary to adopt an approach based on the idea of decoupling the equations of motion.

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS: Undamped systems

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t)$$

$$K\mathbf{u} = \omega^2 M\mathbf{u} \quad U^T M U = I, \; U^T K U = \Omega$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} \eta_r(t)\mathbf{u}_r = U\boldsymbol{\eta}(t)$$

$$\ddot{\boldsymbol{\eta}}(t) + \Omega\boldsymbol{\eta}(t) = \mathbf{N}(t)$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS

$$\ddot{\boldsymbol{\eta}}(t) + \Omega\boldsymbol{\eta}(t) = \mathbf{N}(t)$$

$$\mathbf{N}(t) = U^T\mathbf{Q}(t)$$

$$\ddot{\eta}_r(t) + \omega_r^2\eta_r(t) = N_r(t),\ r = 1, 2, \ldots, n$$

$$N_r(t) = \mathbf{u}_r^T\mathbf{Q}(t),\ r = 1, 2, \ldots, n$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS

Harmonic excitation

$$\mathbf{Q}(t) = \mathbf{Q}_0 \cos \alpha t$$

$$N_r(t) = \mathbf{u}_r^T \mathbf{Q}_0 \cos \alpha t, \quad r = 1, 2, \ldots, n$$

$$\eta_r(t) = \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2} \cos \alpha t, \quad r = 1, 2, \ldots, n$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} \frac{\mathbf{u}_r^T \mathbf{Q}_0}{\omega_r^2 - \alpha^2} \mathbf{u}_r \cos \alpha t$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS: Transient Vibration

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(t-\tau) \sin \omega_r \tau \, d\tau, \quad r = 1, 2, \ldots, n$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} \left[\frac{\mathbf{u}_r^T}{\omega_r} \int_0^t \mathbf{Q}(t-\tau) \sin \omega_r \tau \, d\tau \right] \mathbf{u}_r$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS:

Systems admitting rigid-body modes

$$\ddot{\eta}_r(t) = N_r(t),\ r = 1, 2, \ldots, i$$

$$\eta_r(t) = \int_0^t \left[\int_0^\tau N_r(\sigma)d\sigma\right] d\tau,\ r = 1, 2, \ldots, i$$

$$\mathbf{q}(t) = \sum_{r=1}^{i} \mathbf{u}_r^T \int_0^t \left[\int_0^\tau \mathbf{Q}(\sigma)d\sigma\right] d\tau + \sum_{r=i+1}^{n} \left[\frac{\mathbf{u}_r^T}{\omega_r} \int_0^t \mathbf{Q}(t-\tau)\sin\omega_r\tau\, d\tau\right]\mathbf{u}_r$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS:
Systems with proportional damping

$$C = \alpha M + \beta K$$

$$U^T C U = U^T(\alpha M + \beta K)U = \alpha U^T M U + \beta U^T K U = \alpha I + \beta \Omega$$

$$\ddot{\boldsymbol{\eta}}(t) + (\alpha I + \beta\Omega)\dot{\boldsymbol{\eta}}(t) + \Omega\boldsymbol{\eta}(t) = \mathbf{N}(t)$$

$$\Omega = \mathrm{diag}(\omega_1^2 \; \omega_2^2 \; \ldots \; \omega_n^2)$$

$$\alpha + \beta\omega_r^2 = 2\zeta_r\omega_r, \; r = 1, 2, \ldots, n$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS:

Harmonic excitation

$$\ddot{\eta}_r(t) + 2\zeta_r\omega_r\dot{\eta}_r(t) + \omega_r^2\eta_r(t) = N_r(t),\ r = 1, 2, \ldots, n$$

$$\mathbf{Q}(t) = \mathbf{Q}_0 e^{i\alpha t}$$

$$N_r(t) = \mathbf{u}_r^T\mathbf{Q}_0 e^{i\alpha t},\ r = 1, 2, \ldots, n$$

$$\eta_r(t) = \frac{\mathbf{u}_r^T\mathbf{Q}_0}{\omega_r^2 - \alpha^2 + i2\zeta_r\omega_r\alpha} e^{i\alpha t}$$

$$\mathbf{q}(t) = \sum_{r=1}^{n} \frac{\mathbf{u}_r^T\mathbf{Q}_0}{\omega_r^2 - \alpha^2 + i2\zeta_r\omega_r\alpha} \mathbf{u}_r e^{i\alpha t}$$

7.15 RESPONSE TO EXTERNAL EXCITATIONS BY MODAL ANALYSIS: Transient Vibration

$$\eta_r(t) = \frac{1}{\omega_{dr}} \int_0^t N_r(t-\tau) e^{-\zeta_r \omega_r \tau} \sin \omega_{dr} \tau \, d\tau,$$

$$\omega_{dr} = (1-\zeta_r^2)^{1/2} \omega_r, \; r = 1, 2, \ldots, n$$

7.Multi-Degree-of-Freedom Systems

7.1 Equations of Motion for Linear Systems

7.2 Flexibility and Stiffness Influence Coefficients

7.3 Properties of the Stiffness and Mass Coefficients

7.4 Lagrange's Equations Linearized about Equilibrium

7.5 Linear Transformations : Coupling

7.6 Undamped Free Vibration :The Eigenvalue Problem

7.7 Orthogonality of Modal Vectors

7.8 Systems Admitting Rigid-Body Motions

7.9 Decomposition of the Response in Terms of Modal Vectors

7.10 Response to Initial Excitations by Modal Analysis

7.11 Eigenvalue Problem in Terms of a Single Symmetric Matrix

7.12 Geometric Interpretation of the Eigenvalue Problem

7.13 Rayleigh's Quotient and Its Properties

7.14 Response to Harmonic External Excitations

7.15 Response to External Excitations by Modal Analysis

- **7.15.1** Undamped systems
- **7.15.2** Systems with proportional damping

7.16 Systems with Arbitrary Viscous Damping

7.17 Discrete-Time Systems

Advanced Vibrations

Lecture 9

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7.Multi-Degree-of-Freedom Systems

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$\dot{\mathbf{q}}(t) = \dot{\mathbf{q}}(t)$$

$$\ddot{\mathbf{q}}(t) = -M^{-1}C\dot{\mathbf{q}}(t) - M^{-1}K\mathbf{q}(t) + M^{-1}\mathbf{Q}(t)$$

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{Q}(t)$$

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right], \quad B = \left[\begin{array}{c} 0 \\ \hline M^{-1} \end{array}\right]$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$\mathbf{Q}(t) = \mathbf{0}$$

Nonsymmetric

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$$

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{x}$$

$$A\mathbf{x} = \lambda\mathbf{x} \qquad A^T\mathbf{y} = \lambda\mathbf{y}$$

$$\det[A - \lambda I] = \det[A^T - \lambda I] = 0$$

The eigenvalues/vectors are in general complex.

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING: *Orthogonality*

$$\mathbf{y}^T A = \lambda \mathbf{y}^T \qquad \qquad \mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

Left eigenvectors *Right eigenvectors*

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i\,, \; i = 1, 2, \ldots, 2n$$

$$\mathbf{y}_j^T A = \lambda_j \mathbf{y}_j^T\,, \; j = 1, 2, \ldots, 2n$$

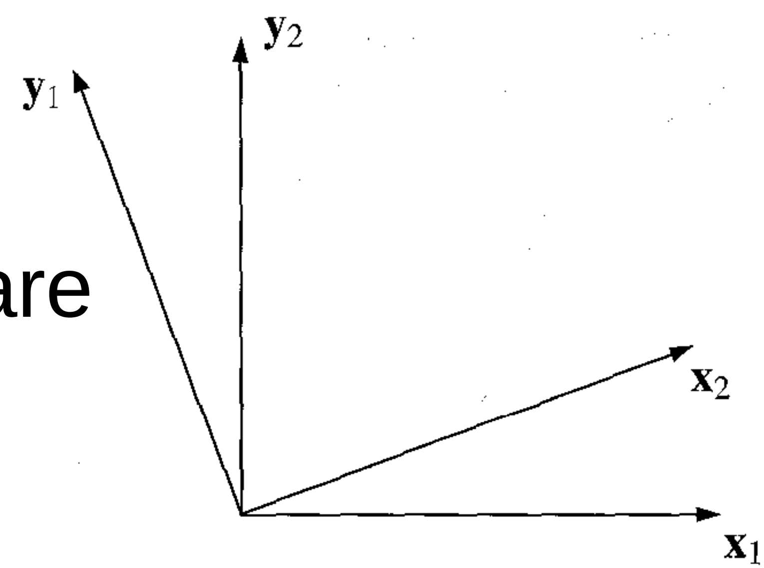
$$\begin{aligned} \mathbf{y}_j^T A\mathbf{x}_i &= \lambda_i \mathbf{y}_j^T \mathbf{x}_i \\ \mathbf{y}_j^T A\mathbf{x}_i &= \lambda_j \mathbf{y}_j^T \mathbf{x}_i \end{aligned} \quad \longrightarrow \quad (\lambda_i - \lambda_j)\mathbf{y}_j^T \mathbf{x}_i = 0$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$(\lambda_i - \lambda_j)\mathbf{y}_j^T \mathbf{x}_i = 0 \Longrightarrow$$

$$\mathbf{y}_j^T \mathbf{x}_i = 0,$$
$$\mathbf{y}_j^T A \mathbf{x}_i = 0,$$
$$\lambda_i \neq \lambda_j, \quad i, j = 1, 2, \ldots, 2n$$

The right eigenvectors $\mathbf{x}_i$ are ***biorthogonal*** to the left eigenvectors $\mathbf{y}_j$.



7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

Biorthonormality Relations

$$\mathbf{y}_j^T \mathbf{x}_i = \delta_{ij} \quad \mathbf{y}_j^T A\mathbf{x}_i = \lambda_i \delta_{ij}, \quad i, j = 1, 2, \ldots, 2n$$

$$Y^T X = I \qquad AX = X\Lambda \qquad Y^T A X = \Lambda$$

$$Y^T = X^{-1} \qquad A^T Y = Y\Lambda \qquad A = X\Lambda Y^T$$

$$XY^T = I$$

The **bi-orthogonality** property forms **the basis for a modal analysis** for the response of systems with arbitrary viscous damping.

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

Assume an arbitrary 2n-dimensional state vector:

$$\mathbf{v} = X\mathbf{a}$$

$$\mathbf{a} = Y^T\mathbf{v} \qquad \Lambda\mathbf{a} = Y^T A\mathbf{v}$$

The expansion theorem forms the basis for a state space modal analysis:

$$\mathbf{x}(t) = \xi_1(t)\mathbf{x}_1 + \xi_2(t)\mathbf{x}_2 + \ldots + \xi_{2n}(t)\mathbf{x}_{2n} = \sum_{r=1}^{2n} \xi_r(t)\mathbf{x}_r$$

$$= X\boldsymbol{\xi}(t)$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{Q}(t)$$

$$Y^T X\dot{\boldsymbol{\xi}}(t) = Y^T AX\boldsymbol{\xi}(t) + Y^T B\mathbf{Q}(t)$$

$$\dot{\boldsymbol{\xi}}(t) = \Lambda\boldsymbol{\xi}(t) + \mathbf{n}(t)$$

$$\mathbf{n}(t) = Y^T B\mathbf{Q}(t)$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING: Harmonic Excitations

$$\mathbf{Q}(t) = \mathbf{Q}_0 e^{i\alpha t}$$

$$n_r(t) = \mathbf{y}_r^T B \mathbf{Q}_0 e^{i\alpha t}$$

$$\xi_r(t) = \Xi_r(i\alpha) e^{i\alpha t}, \qquad r = 1, 2, \ldots, 2n$$

$$\dot{\xi}_r(t) = \lambda_r \xi_r(t) + n_r(t),$$

$$(i\alpha - \lambda_r)\Xi_r(i\alpha) e^{i\alpha t} = \mathbf{y}_r^T B \mathbf{Q}_0 e^{i\alpha t},$$

$$\Xi_r(i\alpha) = \frac{\mathbf{y}_r^T B \mathbf{Q}_0}{i\alpha - \lambda_r}$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING: Harmonic Excitations

$$\xi_r(t) = \frac{\mathbf{Y}_r^T B \mathbf{Q}_0}{i\alpha - \lambda_r} e^{i\alpha t}$$

$$\mathbf{x}(t) = \sum_{r=1}^{2n} \frac{\mathbf{y}_r^T B \mathbf{Q}_0}{i\alpha - \lambda_r} \mathbf{x}_r e^{i\alpha t}$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING: Arbitrary Excitations

$$\dot{\xi}_r(t) = \lambda_r \xi_r(t) + n_r(t),$$

$$s\,\Xi_r(s) - \xi_r(0) = \lambda_r\,\Xi_r(s) + N_r(s),$$

$$\xi_r(0) = \mathbf{y}_r^T \mathbf{x}(0), \; r = 1, 2, \ldots, 2n$$

$$\xi_r(t) = \mathcal{L}^{-1}\,\Xi_r(s) = e^{\lambda_r t}\xi_r(0) + \int_0^t e^{\lambda_r (t-\tau)} n_r(\tau)\,d\tau,$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$\boldsymbol{\xi}(t) = e^{\Lambda t}\boldsymbol{\xi}(0) + \int_0^t e^{\Lambda(t-\tau)}\mathbf{n}(\tau)d\tau$$

$$\mathbf{n}(t) = Y^T B\mathbf{Q}(t) \qquad \boldsymbol{\xi}(0) = Y^T\mathbf{x}(0)$$

$$\mathbf{x}(t) = Xe^{\Lambda t}\boldsymbol{\xi}(0) + \int_0^t Xe^{\Lambda(t-\tau)}\mathbf{n}(\tau)d\tau$$

$$\mathbf{x}(t) = Xe^{\Lambda t}Y^T\mathbf{x}(0) + \int_0^t Xe^{\Lambda(t-\tau)}Y^T B\mathbf{Q}(\tau)d\tau$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$e^{\Lambda t} = I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \frac{t^3}{3!}\Lambda^3 + \dots$$

$$Xe^{\Lambda t}Y^T = XY^T + tX\Lambda Y^T + \frac{t^2}{2!}X\Lambda Y^T X\Lambda Y^T$$

$$+ \frac{t^3}{3!}X\Lambda Y^T X\Lambda Y^T X\Lambda Y^T + \dots$$

$$= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \dots = e^{At}$$

7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$\mathbf{x}(t) = Xe^{\Lambda t}Y^T\mathbf{x}(0) + \int_0^t Xe^{\Lambda(t-\tau)}Y^T B\mathbf{Q}(\tau)d\tau$$

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\,B\mathbf{F}(\tau)d\tau$$

The *state transition matrix*

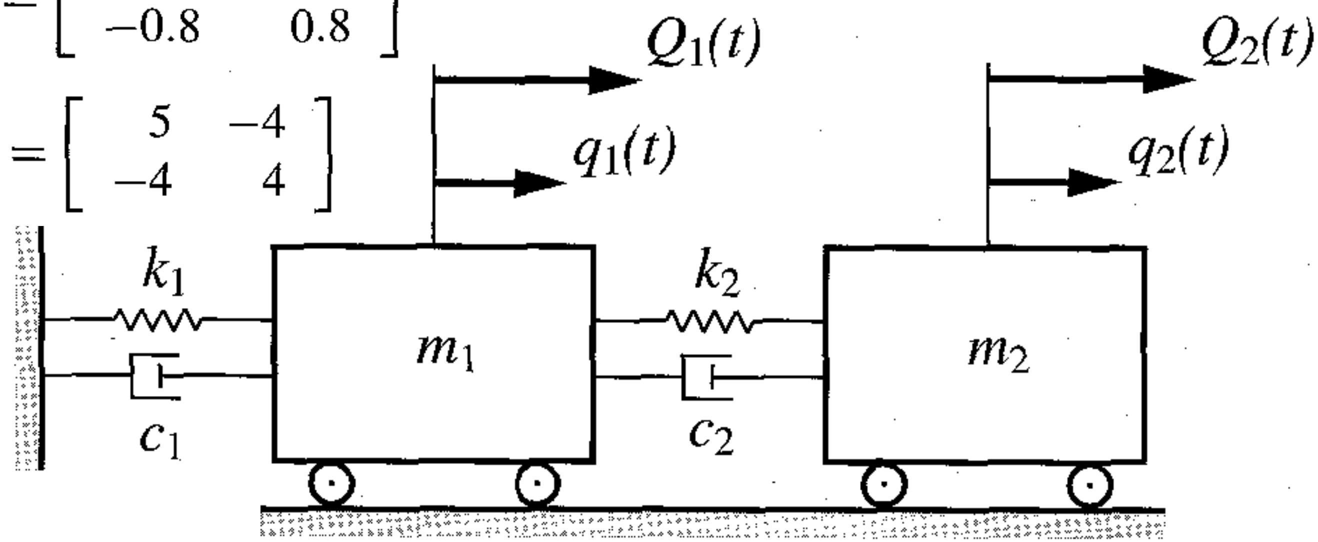
7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

Example 7.12. Determine the response of the system to the excitation: $Q_1(t) = 0,\ Q_2(t) = Q_0[t\mathcal{U}(t) - (t-4)\mathcal{U}(t-4)]$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} 1.6 & -0.8 \\ -0.8 & 0.8 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 4 \end{bmatrix}$$

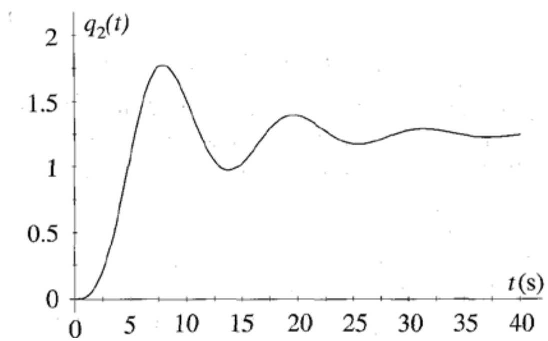


7.16 SYSTEMS WITH ARBITRARY VISCOUS DAMPING

$$\mathbf{x}(t) = [q_1(t)\ q_2(t)\ \dot{q}_1(t)\ \dot{q}_2(t)]^T \qquad \mathbf{x}(0) = \mathbf{0}.$$

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}K & -M^{-1}C \end{array}\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 4 & -1.6 & 0.8 \\ 2 & -2 & 0.4 & -0.4 \end{bmatrix}$$

$$B = \left[\begin{array}{c} 0 \\ \hline M^{-1} \end{array}\right] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$



$$\mathbf{x}(t) = e^{At}\,\mathbf{x}(0) + \int_0^t e^{A(t-\tau)}\,B\mathbf{F}(\tau)\,d\tau$$

7.Multi-Degree-of-Freedom Systems