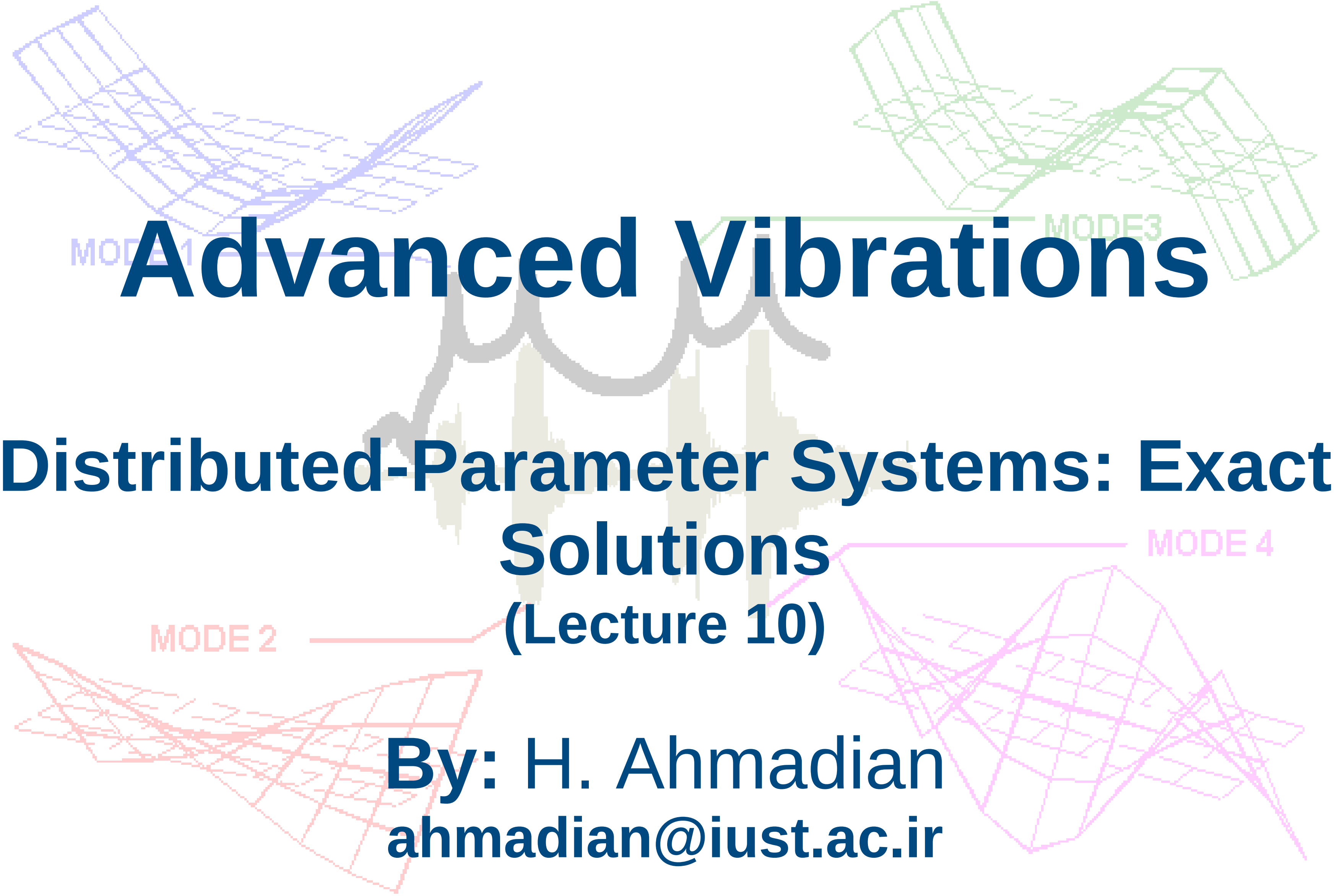

UMASS LOWELL MODAL ANALYSIS and CONTROLS LABORATORY - Pete Avitabile and Fabio Piergentili

Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 10)

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Distributed-Parameter Systems: Exact Solutions

- Relation between Discrete and Distributed Systems .
- Transverse Vibration of Strings
- Derivation of the String Vibration Problem by the Extended Hamilton Principle
- Bending Vibration of Beams
- Free Vibration: The Differential Eigenvalue Problem
- Orthogonality of Modes Expansion Theorem
- Systems with Lumped Masses at the Boundaries
- Eigenvalue Problem and Expansion Theorem for Problems with Lumped Masses at the Boundaries
- Rayleigh's Quotient . The Variational Approach to the Differential Eigenvalue Problem
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- Traveling Waves in Rods of Finite Length

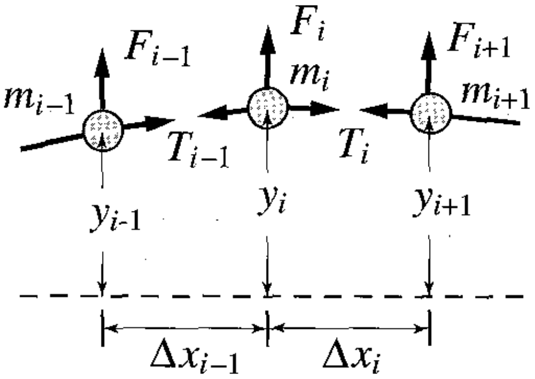
Introduction

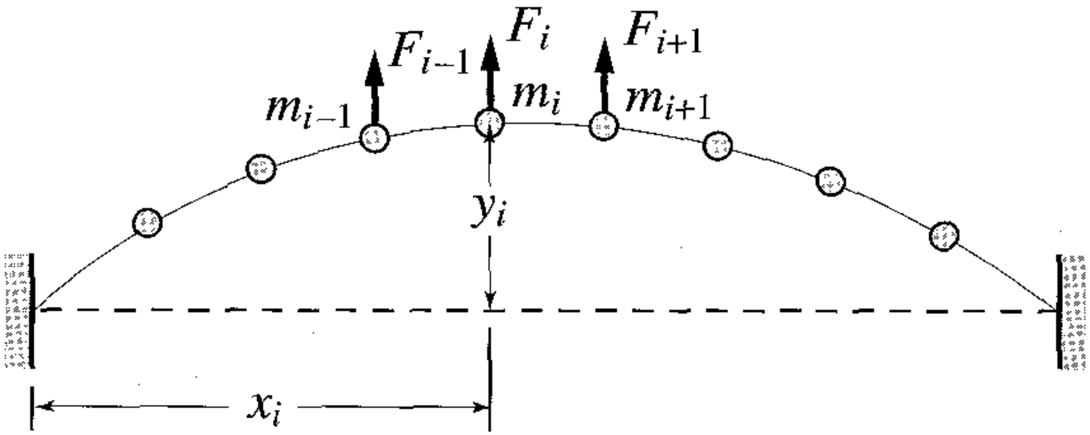
- The motion of distributed-parameter systems is governed by partial differential equations:
  - to be satisfied over the domain of the system, and
  - is subject to boundary conditions at the end points of the domain.
- Such problems are known as *boundary-value problems.*

RELATION BETWEEN DISCRETE AND DISTRIBUTED SYSTEMS: TRANSVERSE VIBRATION OF STRINGS

$$T_i \frac{y_{i+1} - y_i}{\Delta x_i} - T_{i-1} \frac{y_i - y_{i-1}}{\Delta x_{i-1}} + F_i = m_i \frac{d^2 y_i}{dt^2} \qquad i = 1, 2, \ldots, n$$

$$\frac{T_i}{\Delta x_i} y_{i+1} - \left(\frac{T_i}{\Delta x_i} + \frac{T_{i-1}}{\Delta x_{i-1}} \right) y_i + \frac{T_{i-1}}{\Delta x_{i-1}} y_{i-1} + F_i = m_i \frac{d^2 y_i}{dt^2},$$





RELATION BETWEEN DISCRETE AND DISTRIBUTED SYSTEMS: TRANSVERSE VIBRATION OF STRINGS

$$y_{i+1} - y_i = \Delta y_i, \ \ y_i - y_{i-1} = \Delta y_{i-1}$$

$$T_i \frac{\Delta y_i}{\Delta x_i} - T_{i-1} \frac{\Delta y_{i-1}}{\Delta x_{i-1}} + F_i = m_i \frac{d^2 y_i}{dt^2},$$

$$\Delta \left(T_i \frac{\Delta y_i}{\Delta x_i} \right) + F_i = m_i \frac{d^2 y_i}{dt^2}$$

$$\frac{\Delta}{\Delta x_i} \left(T_i \frac{\Delta y_i}{\Delta x_i} \right) + \frac{F_i}{\Delta x_i} = \frac{m_i}{\Delta x_i} \frac{d^2 y_i}{dt^2},$$

partial differential equation of motion of the string

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial y(x,t)}{\partial x} \right] + f(x,t) = \rho(x) \frac{\partial^2 y(x,t)}{\partial t^2}$$

$$f(x,t) = \lim_{\Delta x_i \to 0} \frac{F_i(t)}{\Delta x_i}, \ \ \rho(x) = \lim_{\Delta x_i \to 0} \frac{m_i}{\Delta x_i}$$

RELATION BETWEEN DISCRETE AND DISTRIBUTED SYSTEMS: TRANSVERSE VIBRATION OF STRINGS

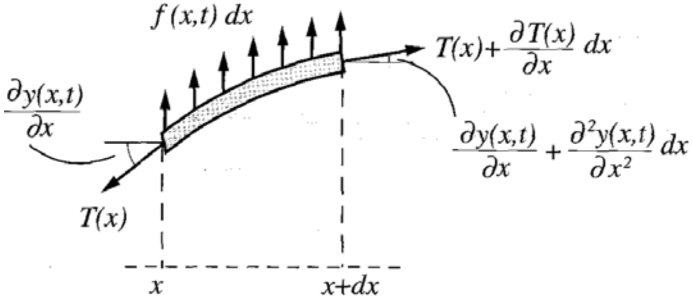
$$\left[T(x)+\frac{\partial T(x)}{\partial x}dx\right]\left[\frac{\partial y(x,t)}{\partial x}+\frac{\partial^2 y(x,t)}{\partial x^2}dx\right]$$

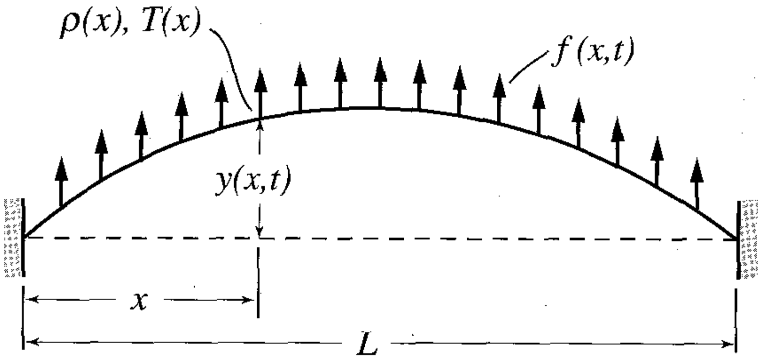
$$-T(x)\frac{\partial y(x,t)}{\partial x}+f(x,t)dx=\rho(x)dx\frac{\partial^2 y(x,t)}{\partial t^2}$$

Ignoring 2nd order term

$$\frac{\partial T(x)}{\partial x}\frac{\partial y(x,t)}{\partial x}dx+T(x)\frac{\partial^2 y(x,t)}{\partial x^2}dx+f(x,t)dx=\rho(x)dx\frac{\partial^2 y(x,t)}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]+f(x,t)=\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}$$



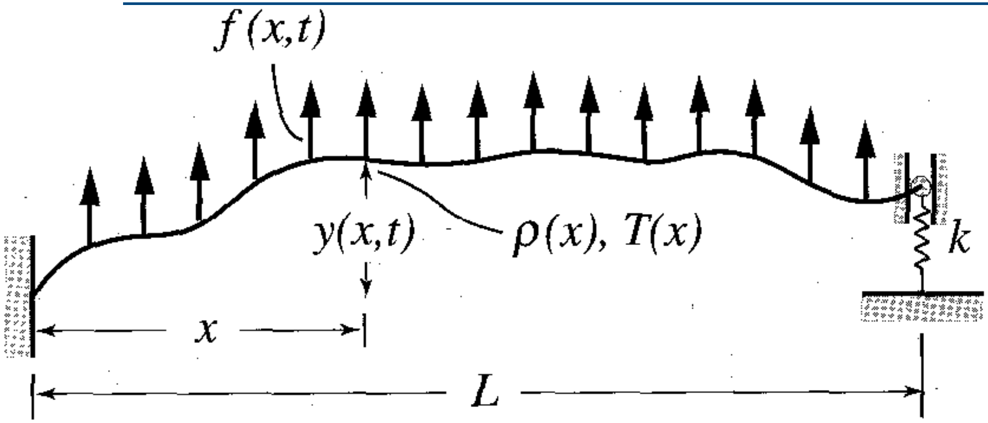


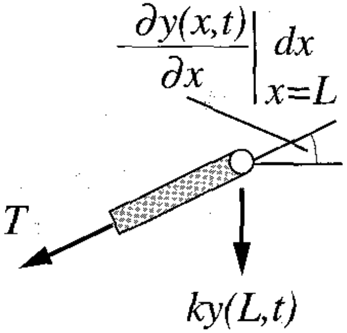
DERIVATION OF THE STRING VIBRATION PROBLEM BY THE EXTENDED HAMILTON PRINCIPLE

$$\int_{t_1}^{t_2}(\delta T-\delta V+\overline{\delta W}_{nc})dt=0,\ \delta y(x,t)=0,\quad 0\le x\le L,\ t=t_1,t_2$$

$$T(t)=\frac{1}{2}\int_0^L\rho(x)\left[\frac{\partial y(x,t)}{\partial t}\right]^2dx\qquad\overline{\delta W}_{nc}(t)=\int_0^L f(x,t)\delta y(x,t)dx$$

$$V(t)=\int_0^L T(x)[ds(x,t)-dx]+\tfrac{1}{2}ky^2(L,t)$$



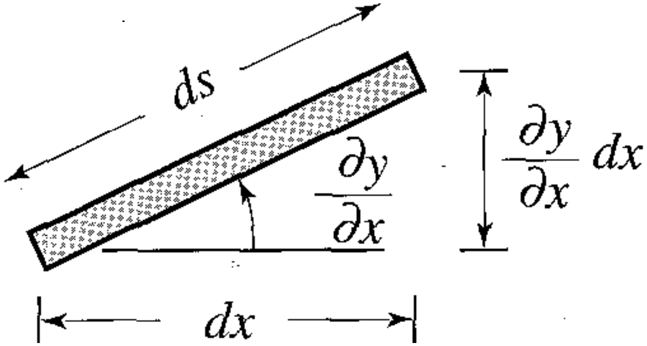


DERIVATION OF THE STRING VIBRATION PROBLEM BY THE EXTENDED HAMILTON PRINCIPLE

$$V(t) = \int_0^L T(x)[ds(x,t) - dx] + \tfrac{1}{2}ky^2(L,t)$$

$$ds = \left[(dx)^2 + \left(\frac{\partial y}{\partial x}dx\right)^2\right]^{1/2} = \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} dx \cong \left[1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right] dx$$

$$V(t) = \frac{1}{2}\int_0^L T(x)\left[\frac{\partial y(x,t)}{\partial x}\right]^2 dx + \tfrac{1}{2}ky^2(L,t)$$



DERIVATION OF THE STRING VIBRATION PROBLEM BY THE EXTENDED HAMILTON PRINCIPLE

$$\delta T = \int_0^L \rho \frac{\partial y}{\partial t} \delta \left(\frac{\partial y}{\partial t} \right) dx = \int_0^L \rho \frac{\partial y}{dt} \frac{\partial}{\partial t} \delta y \, dx$$

$$\int_{t_1}^{t_2} \delta T \, dt = \int_{t_1}^{t_2} \left(\int_0^L \rho \frac{\partial y}{\partial t} \frac{\partial}{\partial t} \delta y dx \right) dt = \int_0^L \left(\int_{t_1}^{t_2} \rho \frac{\partial y}{\partial t} \frac{\partial}{\partial t} \delta y \, dt \right) dx$$

$$= \int_0^L \left(\rho \frac{\partial y}{\partial t} \delta y \Big|_{t_1}^{t_2} \right) dx - \int_0^L \left(\int_{t_1}^{t_2} \rho \frac{\partial^2 y}{\partial t^2} \delta y dt \right) dx$$

$$= - \int_{t_1}^{t_2} \left(\int_0^L \rho \frac{\partial^2 y}{\partial t^2} \delta y \, dx \right) dt$$

DERIVATION OF THE STRING VIBRATION PROBLEM BY THE EXTENDED HAMILTON PRINCIPLE

$$\delta V = \int_0^L T\frac{\partial y}{\partial x}\delta\frac{\partial y}{\partial x}dx + ky(L,t)\delta y(L,t)$$

$$= \int_0^L T\frac{\partial y}{\partial x}\frac{\partial}{\partial x}\delta y dx + ky(L,t)\delta y(L,t)$$

$$\delta V = T\frac{\partial y}{\partial x}\delta y\Big|_0^L - \int_0^L \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\delta y dx + ky(L,t)\delta y(L,t)$$

$$= \left(T\frac{\partial y}{\partial x} + ky\right)\delta y\Big|_{x=L} - T\frac{\partial y}{\partial x}\delta y\Big|_{x=0} - \int_0^L \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)\delta y dx$$

DERIVATION OF THE STRING VIBRATION PROBLEM BY THE EXTENDED HAMILTON PRINCIPLE

$$\int_{t_1}^{t_2}\left\{\int_0^L\left[-\rho\frac{\partial^2 y}{\partial t^2}+\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)+f\right]\delta y\,dx\right.$$

$$\left.-\left(T\frac{\partial y}{\partial x}+ky\right)\delta y\Big|_{x=L}+T\frac{\partial y}{\partial x}\delta y\Big|_{x=0}\right\}dt=0$$

EOM

$$\frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right)+f=\rho\frac{\partial^2 y}{\partial t^2},\quad 0<x<L$$

BC's

$$T\frac{\partial y}{\partial x}\delta y=0,\quad x=0$$

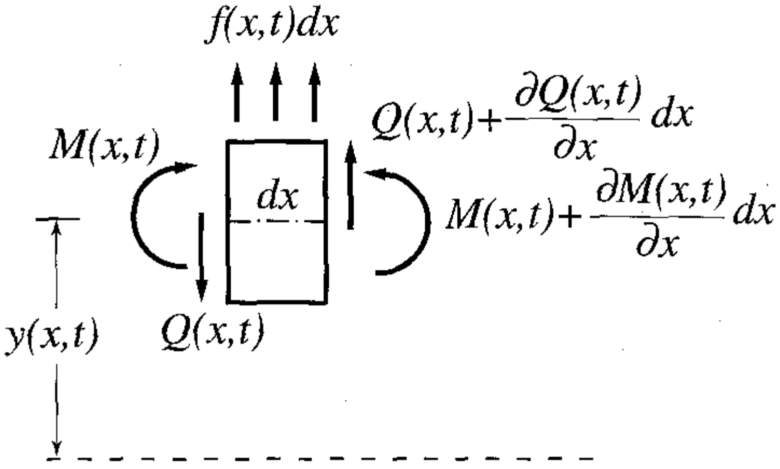
$$\left(T\frac{\partial y}{\partial x}+ky\right)\delta y=0,\quad x=L$$

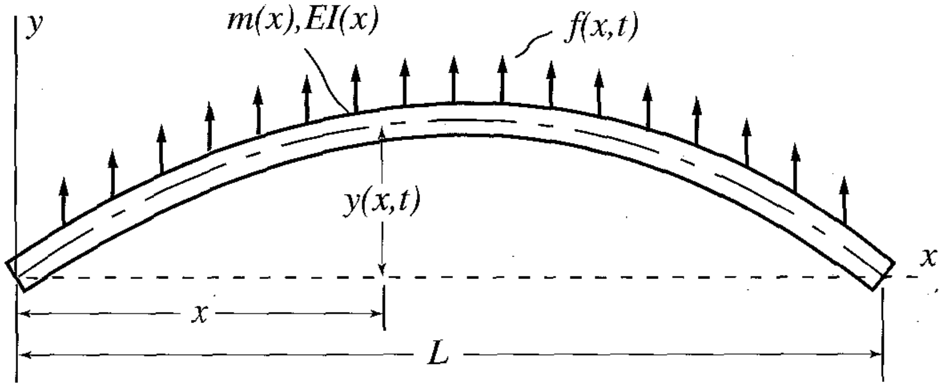
BENDING VIBRATION OF BEAMS

$$\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]-Q(x,t)+f(x,t)dx=m(x)dx\frac{\partial^2 y(x,t)}{\partial t^2},$$

$$\left[M(x,t)+\frac{\partial M(x,t)}{\partial x}dx\right]-M(x,t)+\left[Q(x,t)+\frac{\partial Q(x,t)}{\partial x}dx\right]dx$$

$$+f(x,t)dx\frac{dx}{2}=0,\ 0<x<L$$





BENDING VIBRATION OF BEAMS

$$-\frac{\partial^2 M(x,t)}{\partial x^2}+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},$$

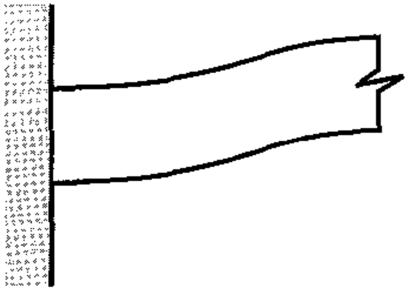
$$\frac{\partial M(x,t)}{\partial x}+Q(x,t)=0,$$

$$M(x,t)=EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}$$

$$Q(x,t)=-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]$$

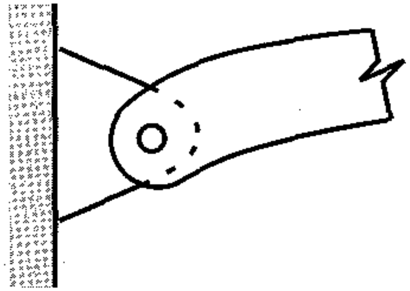
$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\quad 0<x<L$$

BENDING VIBRATION OF BEAMS



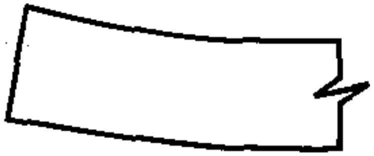
$$y(x,t)=0,$$

$$\frac{\partial y(x,t)}{\partial x}=0$$



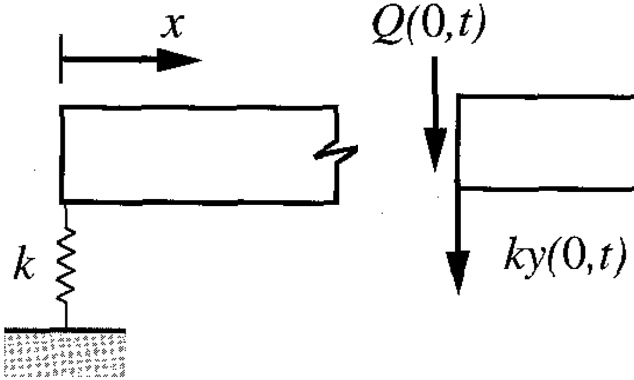
$$y(x,t)=0,$$

$$\frac{\partial^2 y(x,t)}{\partial x^2}=0$$



$$\frac{\partial^2 y(x,t)}{\partial x^2}=0,$$

$$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=0$$



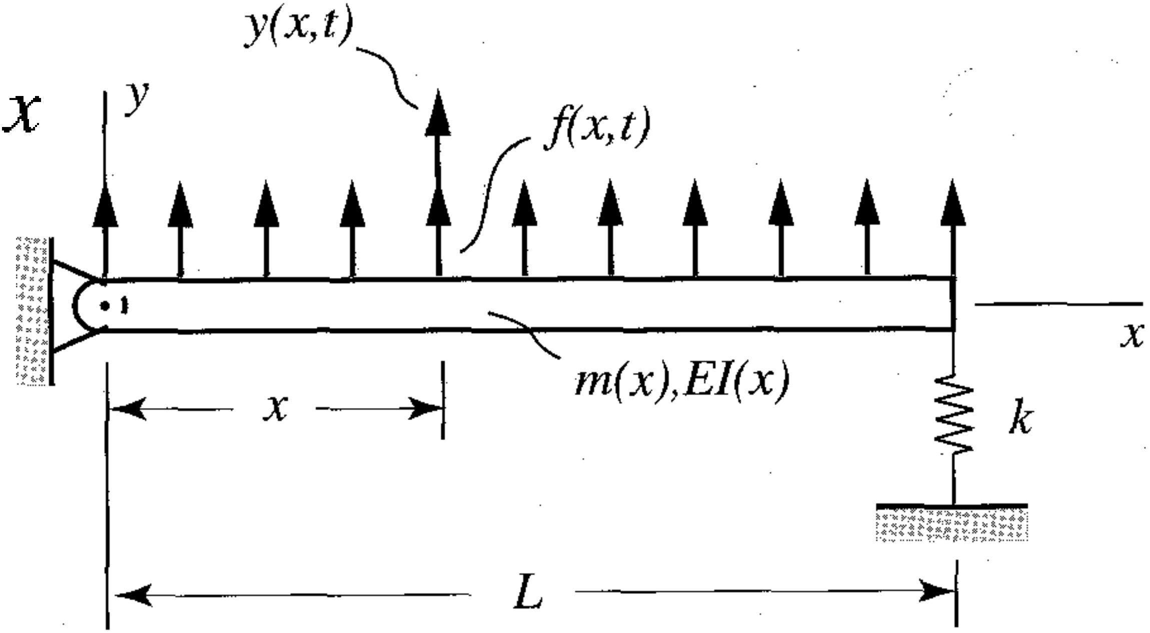
$$Q(x,t)=-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=ky(x,t),\ x=0$$

BENDING VIBRATION OF BEAMS:EHP

$$V(t) = \frac{1}{2}\int_0^L EI(x)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]^2 dx + \tfrac{1}{2}ky^2(L,t)$$

$$\int_{t_1}^{t_2} \delta T\,dt = -\int_{t_1}^{t_2}\left(\int_0^L m\frac{\partial^2 y}{\partial t^2}\delta y\,dx\right)dt$$

$$\overline{\delta W}_{nc} = \int_0^L f\,\delta y\,dx$$



BENDING VIBRATION OF BEAMS:EHP

$$
\begin{aligned}
\delta V &= \int_0^L EI \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial^2 y}{\partial x^2} dx + ky(L,t)\delta y(L,t) = \int_0^L EI \frac{\partial^2 y}{\partial x^2} \frac{\partial^2}{\partial x^2} \delta y dx + ky(L,t)\delta y(L,t) \\
&= EI \frac{\partial^2 y}{\partial x^2} \frac{\partial}{\partial x} \delta y \Big|_0^L - \frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) \delta y \Big|_0^L + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) \delta y dx + ky(L,t)\delta y(L,t) \\
&= EI \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial y}{\partial x} \Big|_{x=L} - EI \frac{\partial^2 y}{\partial x^2} \delta \frac{\partial y}{\partial x} \Big|_{x=0} - \left[\frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) - ky\right] \delta y \Big|_{x=L} \\
&\quad + \frac{\partial}{\partial x}\left(EI \frac{\partial^2 y}{\partial x^2}\right) \delta y \Big|_{x=0} + \int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) \delta y dx
\end{aligned}
$$

BENDING VIBRATION OF BEAMS:EHP

$$\int_{t_1}^{t_2}\left\{-\int_0^L\left[m\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)-f\right]\delta y\,dx-EI\frac{\partial^2 y}{\partial x^2}\delta\frac{\partial y}{\partial x}\Big|_{x=L}\right.$$

$$\left.+EI\frac{\partial^2 y}{\partial x^2}\delta\frac{\partial y}{\partial x}\Big|_{x=0}+\left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right)-ky\right]\delta y\Big|_{x=L}-\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right)\delta y\Big|_{x=0}\right\}dt=0$$

$$-\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)+f=m\frac{\partial^2 y}{\partial t^2},\ 0<x<L$$

$$EI\frac{\partial^2 y}{\partial x^2}\delta\frac{\partial y}{\partial x}=0,\ \frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right)\delta y=0,\ x=0$$

$$EI\frac{\partial^2 y}{\partial x^2}\delta\frac{\partial y}{\partial x}=0,\ \left[\frac{\partial}{\partial x}\left(EI\frac{\partial^2 y}{\partial x^2}\right)-ky\right]\delta y=0,\ x=L$$

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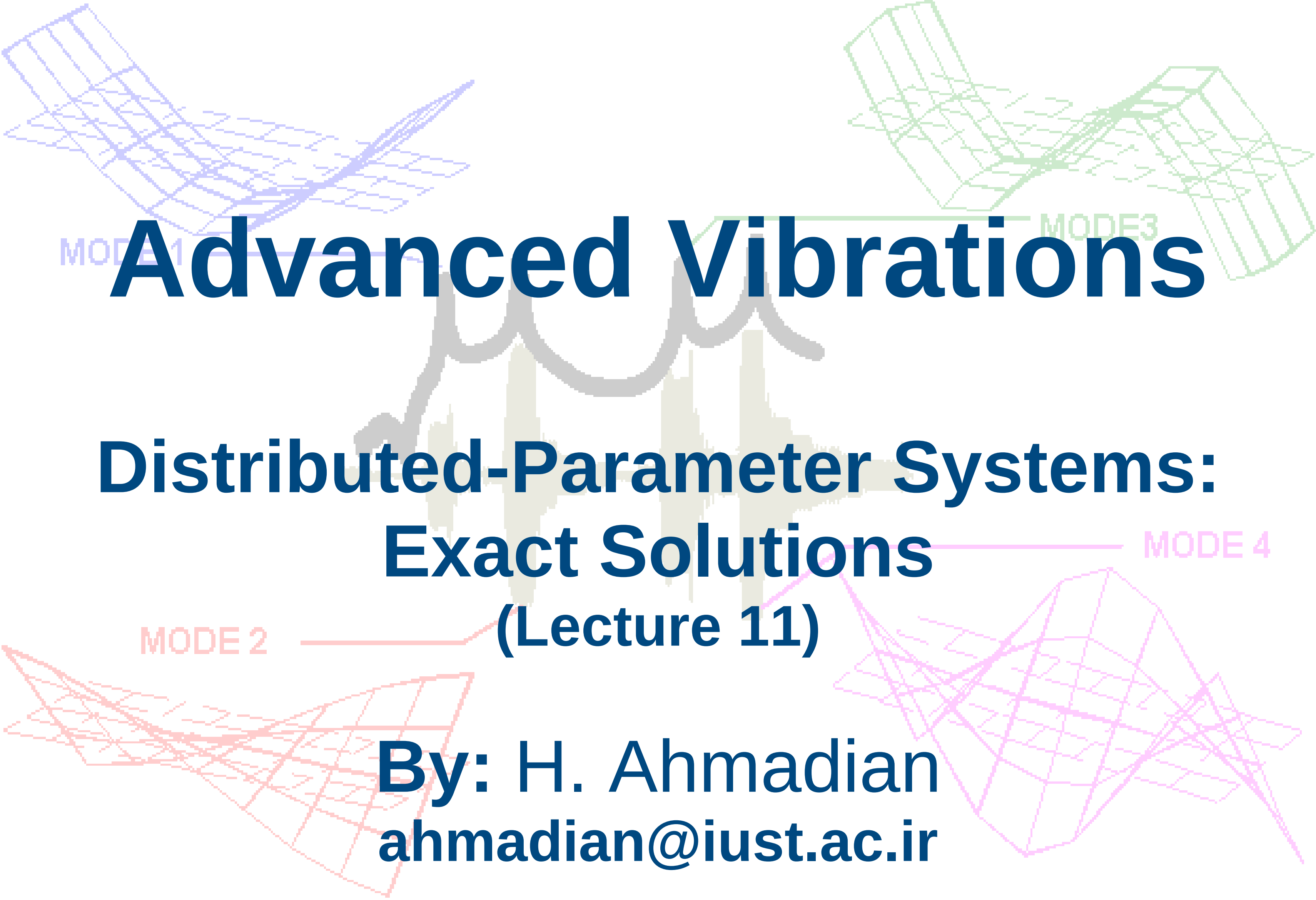

Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 11)

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FREE VIBRATION. THE DIFFERENTIAL EIGENVALUE PROBLEM

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]=\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2},\ 0<x<L$$

$$y(0,t)=0,\ \ y(L,t)=0$$

$$y(x,t)=Y(x)F(t)$$

$$\frac{1}{\rho(x)Y(x)}\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]=\frac{1}{F(t)}\frac{d^2F(t)}{dt^2}=\lambda$$

FREE VIBRATION. THE DIFFERENTIAL EIGENVALUE PROBLEM

$$\frac{d^2F(t)}{dt^2} - \lambda F(t) = 0$$

$$F(t) = Ae^{st} \qquad s^2 - \lambda = 0$$

$$\lambda = -\omega^2 \qquad \begin{matrix} s_1 \\ s_2 \end{matrix} = \pm\sqrt{-\omega^2} = \pm i\omega$$

On physical grounds

$$F(t) = A_1e^{s_1t} + A_2e^{s_2t} = A_1e^{i\omega t} + A_2e^{-i\omega t}$$

$$F(t) = C\cos(\omega t - \phi)$$

FREE VIBRATION. THE DIFFERENTIAL EIGENVALUE PROBLEM

The *differential eigenvalue problem*

$$-\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]=\omega^2\rho(x)Y(x),\ 0<x<L$$

$$Y(0)=0,\ Y(L)=0$$

$$\rho(x)=\rho=\text{constant},\ T(x)=T=\text{constant}$$

$$\frac{d^2Y(x)}{dx^2}+\beta^2Y(x)=0,\ 0<x<L,\ \beta^2=\frac{\omega^2\rho}{T}$$

$$Y(x)=A\sin\beta x+B\cos\beta x$$

FREE VIBRATION. THE DIFFERENTIAL EIGENVALUE PROBLEM

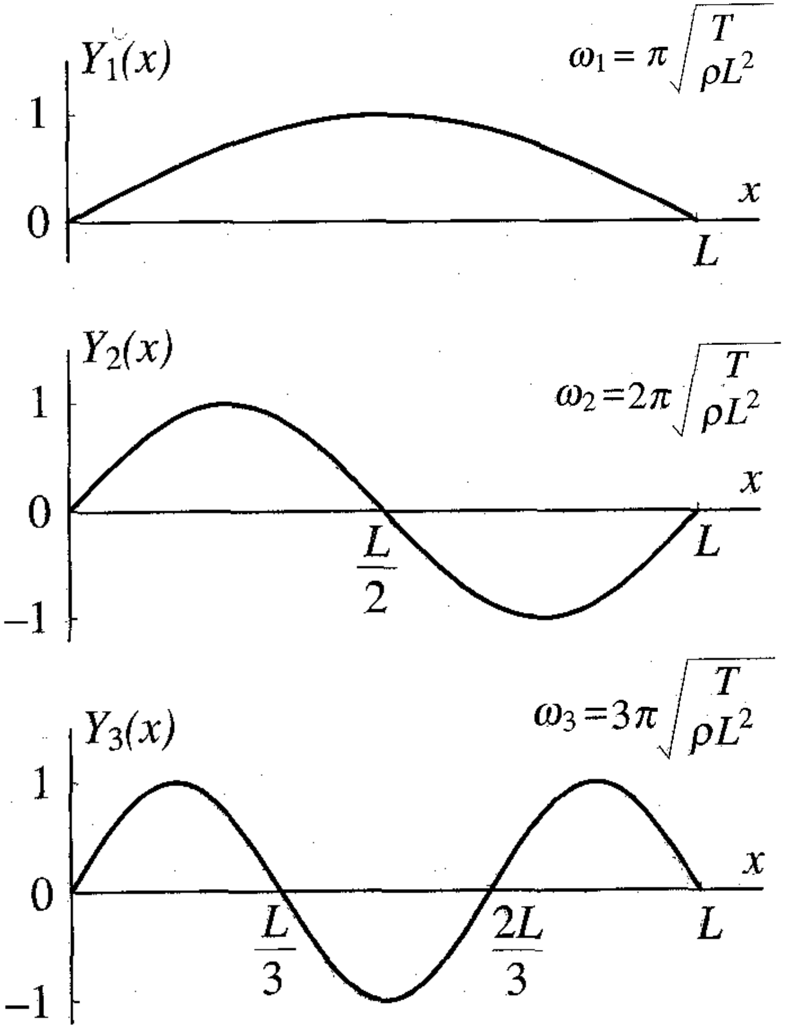
$$Y(0) = 0, \longrightarrow B = 0$$

$$Y(L) = 0 \longrightarrow \sin \beta L = 0$$

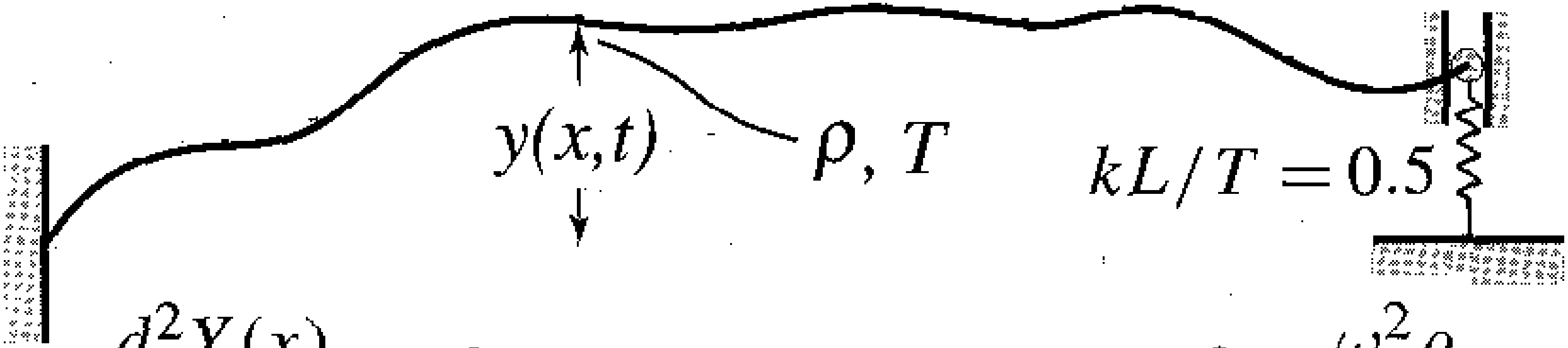
$$\omega_r = \beta_r \sqrt{\frac{T}{\rho}} = r\pi \sqrt{\frac{T}{\rho L^2}},$$

$$Y_r(x) = A_r \sin \frac{r\pi x}{L},$$

$$r = 1, 2, \ldots$$



Example:



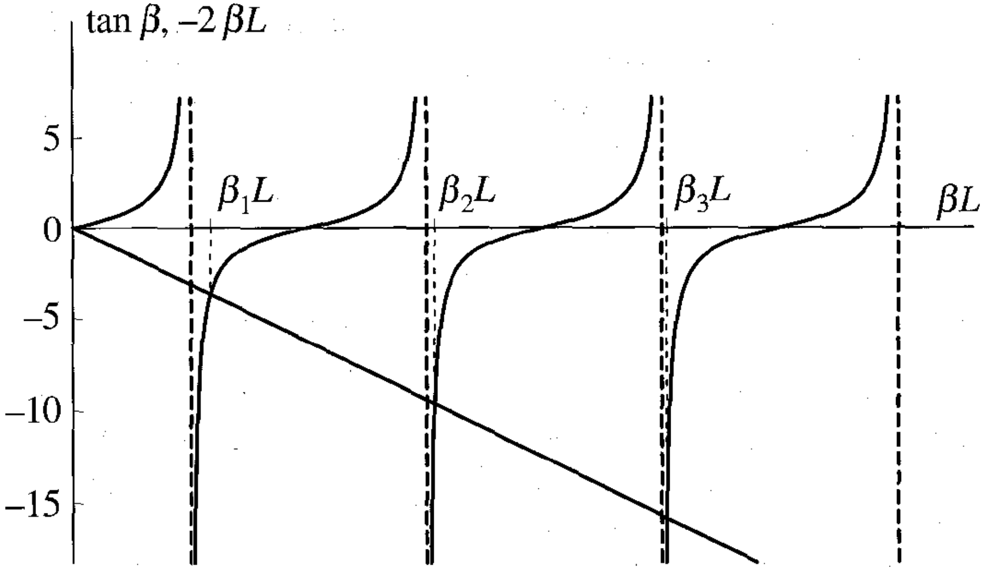
$$\frac{d^2Y(x)}{dx^2} + \beta^2 Y(x) = 0,\ 0 < x < L;\ \beta^2 = \frac{\omega^2 \rho}{T}$$

$$Y(x) = A\sin\beta x + B\cos\beta x$$

$$Y(0) = 0; \qquad Y(x) = A\sin\beta x$$

$$T\frac{dY(x)}{dx} + kY(x) = 0,\ x = L$$

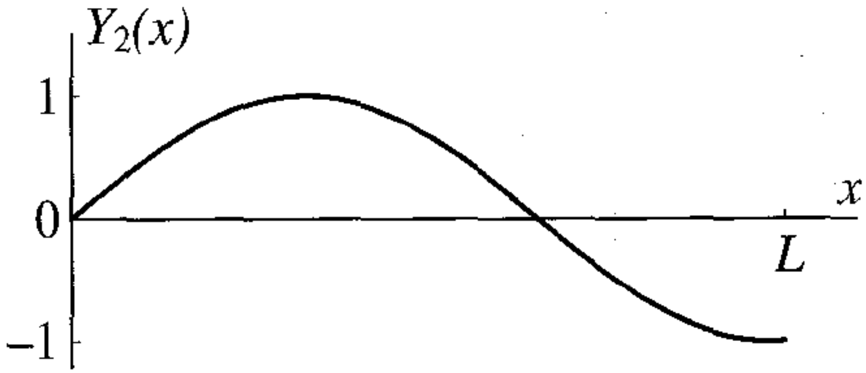
$$\tan\beta L = -\frac{T}{kL}\beta L = -2\beta L$$



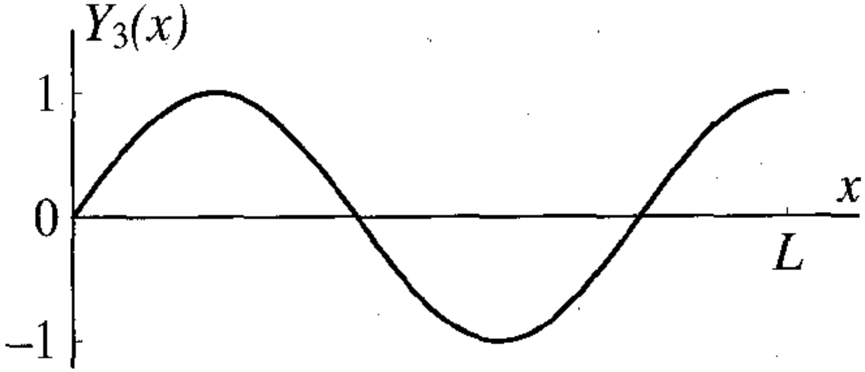
Example:

$$\omega_1 = 1.8366\sqrt{\frac{T}{\rho L^2}}$$

$$\omega_2 = 4.8158\sqrt{\frac{T}{\rho L^2}}$$



$$\omega_3 = 7.9171\sqrt{\frac{T}{\rho L^2}}$$



The free vibration of beams in bending:

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\quad 0<x<L$$

The differential eigenvalue problem:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=\omega^2 m(x)Y(x),$$

$$EI(x)=EI,\ m(x)=m$$

$$\frac{d^4Y(x)}{dx^4}-\beta^4 Y(x)=0,\ 0<x<L;\ \beta^4=\frac{\omega^2 m}{EI}$$

$$Y(x)=A\sin\beta x+B\cos\beta x+C\sinh\beta x+D\cosh\beta x$$

Simply Supported beam:

$$Y(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

$$Y(0) = B + D = 0$$

$$\left.\frac{d^2Y(x)}{dx^2}\right|_{x=0} = -B + D = 0$$

$$B = D = 0$$

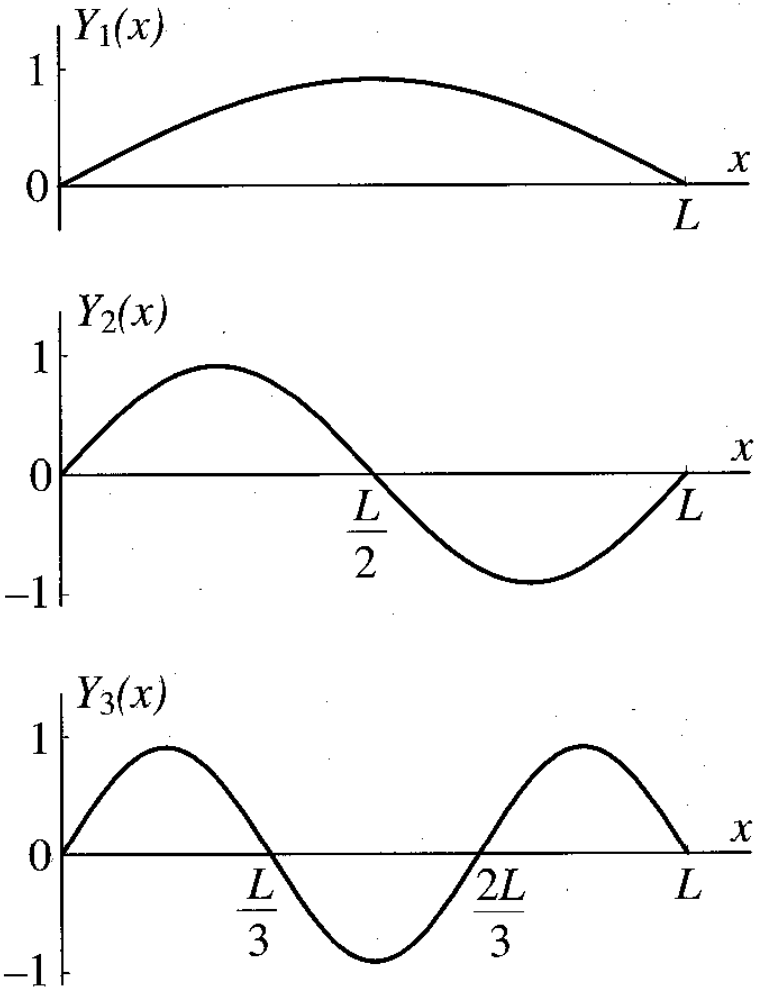
$$Y(L) = A\sin\beta L + C\sinh\beta L = 0$$

$$\frac{d^2Y(x)}{dx^2} = \beta^2(-A\sin\beta L + C\sinh\beta L) = 0$$

$$C = 0 \qquad \sin\beta L = 0$$

$$\beta_r L = r\pi,\ r = 1, 2, \ldots$$

$$Y_r(x) = A_r \sin\frac{r\pi x}{L}$$



Uniform Clamped Beam:

$$Y(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

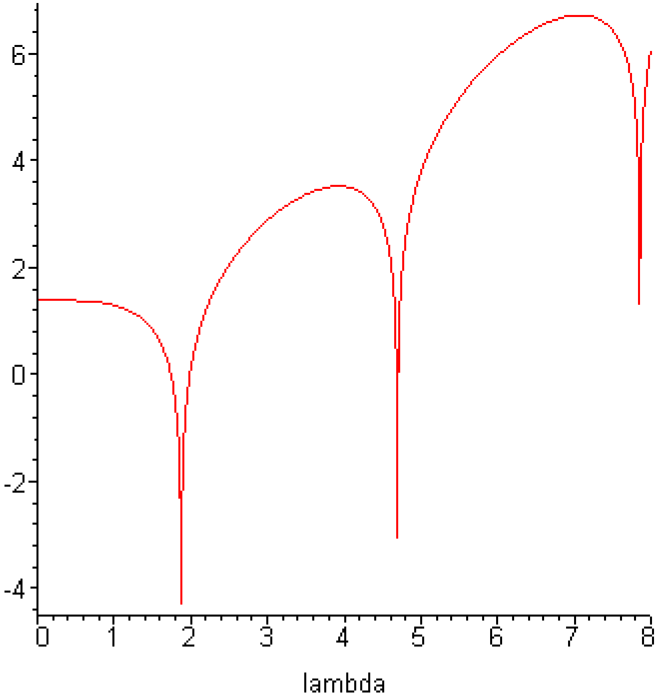
$$Y(x) = 0, \quad \frac{dY(x)}{dx} = 0, \quad x = 0$$

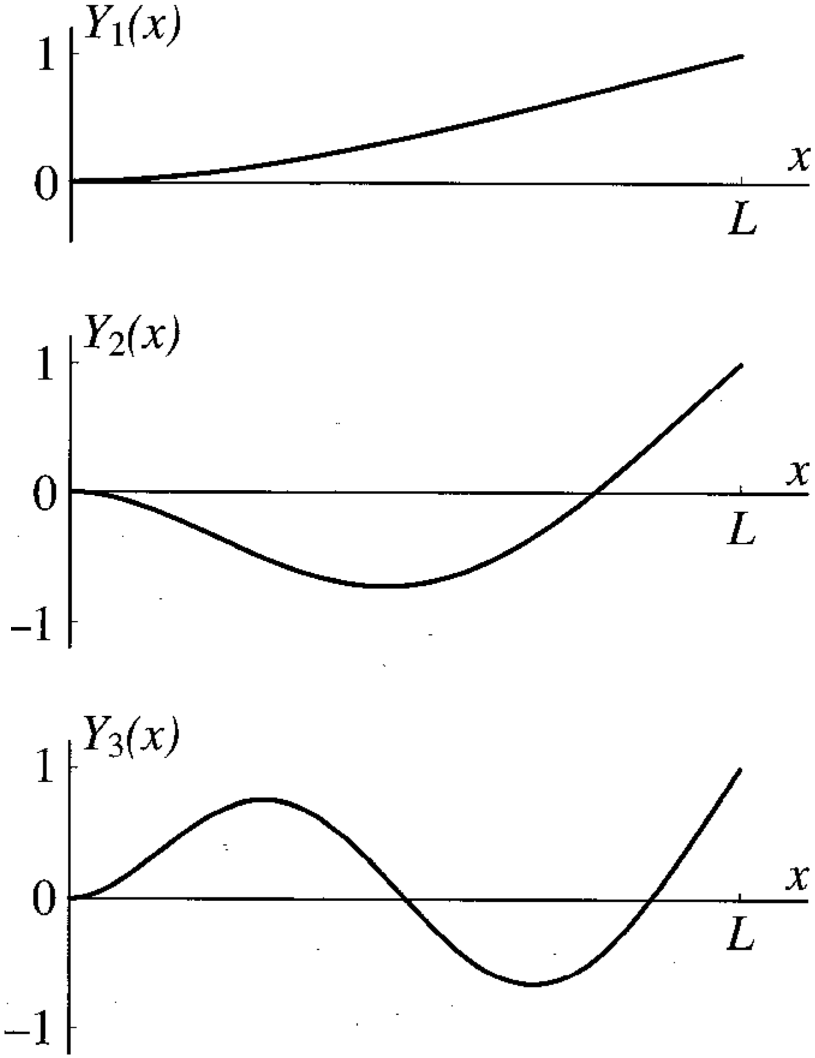
$$\frac{d^2Y(x)}{dx^2} = 0, \quad \frac{d^3Y(x)}{dx^3} = 0, \quad x = L$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(\beta L) & -\cos(\beta L) & \sinh(\beta L) & \cosh(\beta L) \\ -\cos(\beta L) & \sin(\beta L) & \cosh(\beta L) & \sinh(\beta L) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

$$\cos\beta L \cosh\beta L = -1$$

Uniform Clamped Beam:



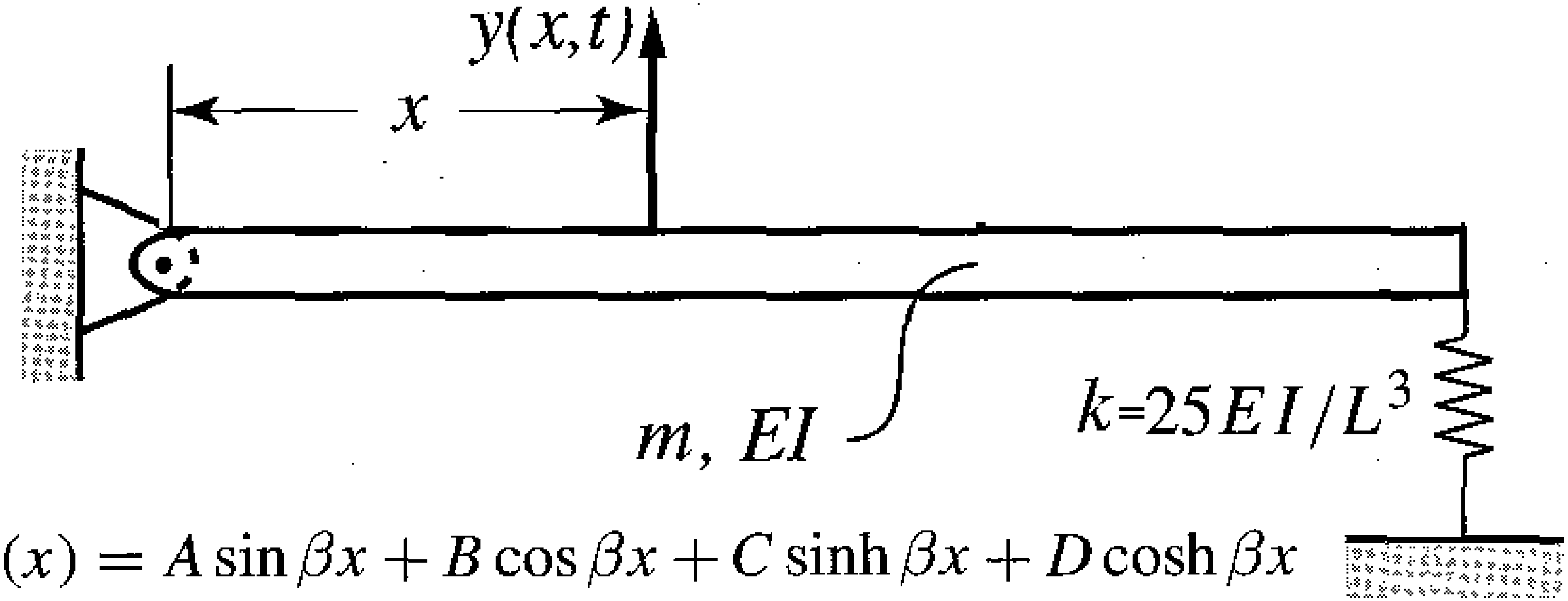


$$\omega_1 = 3.5160\sqrt{\frac{EI}{mL^4}}$$

$$\omega_2 = 22.0345\sqrt{\frac{EI}{mL^4}}$$

$$\omega_3 = 61.6972\sqrt{\frac{EI}{mL^4}}$$

The spring supported-pinned beam



$$Y(x) = A\sin\beta x + B\cos\beta x + C\sinh\beta x + D\cosh\beta x$$

$$Y(x) = 0,\ \frac{d^2Y(x)}{dx^2} = 0,\ x = 0 \Longrightarrow B = D = 0$$

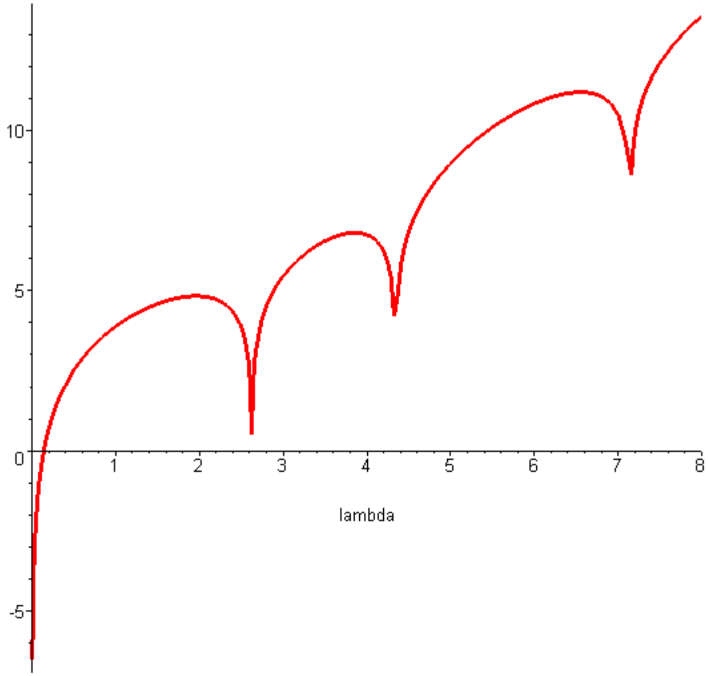
$$\frac{d^2Y(x)}{dx^2} = 0,\ \frac{d^3Y(x)}{dx^3} = \frac{k}{EI}Y(x),\ x = L$$

$\Longrightarrow$ **Characteristic equation**

The spring supported-pinned beam

$$= \begin{bmatrix} -\sin(\lambda) & \sinh(\lambda) \\ 25\sin(\lambda)+\lambda^3\cos(\lambda) & 25\sinh(\lambda)-\lambda^3\cosh(\lambda) \end{bmatrix} \begin{bmatrix} A \\ C \end{bmatrix} := 0$$

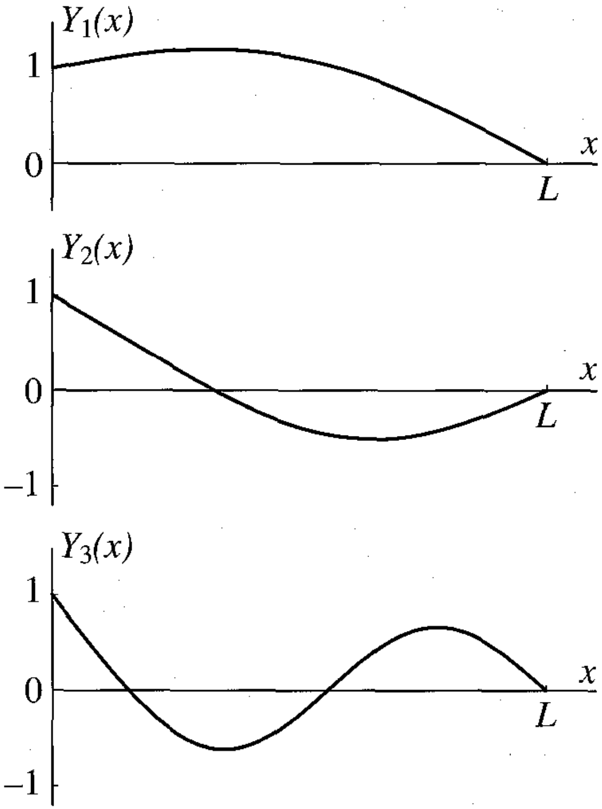
$$\coth\beta L - \cot\beta L = \frac{50}{(\beta L)^3}$$



$$\omega_1 = 6.8858\sqrt{\frac{EI}{mL^4}}$$

$$\omega_2 = 18.8462\sqrt{\frac{EI}{mL^4}}$$

$$\omega_3 = 51.0100\sqrt{\frac{EI}{mL^4}}$$



ORTHOGONALITY OF MODES. EXPANSION THEOREM

Consider two distinct solutions of the string eigenvalue problem:

$$-\frac{d}{dx}\left[T(x)\frac{dY_r(x)}{dx}\right]=\omega_r^2\rho(x)Y_r(x), \qquad -\frac{d}{dx}\left[T(x)\frac{dY_s(x)}{dx}\right]=\omega_s^2\rho(x)Y_s(x),$$

$$\int_0^L T(x)\frac{dY_s(x)}{dx}\frac{dY_r(x)}{dx}dx=\omega_r^2\int_0^L \rho(x)Y_s(x)Y_r(x)dx$$

$$\int_0^L T(x)\frac{dY_r(x)}{dx}\frac{dY_s(x)}{dx}dx=\omega_s^2\int_0^L \rho(x)Y_r(x)Y_s(x)dx$$

$$\int_0^L \rho(x)Y_r(x)Y_s(x)dx=0$$

ORTHOGONALITY OF MODES. EXPANSION THEOREM

$$\int_0^L \rho(x)Y_r^2(x)dx = 1, \quad \int_0^L T(x)\left[\frac{dY_r(x)}{dx}\right]^2 dx = \omega_r^2,$$

$$r, s = 1, 2, \ldots; \; \omega_r^2 \neq \omega_s^2$$

$$\int_0^L \rho(x)Y_r(x)Y_s(x)dx = \delta_{rs},$$

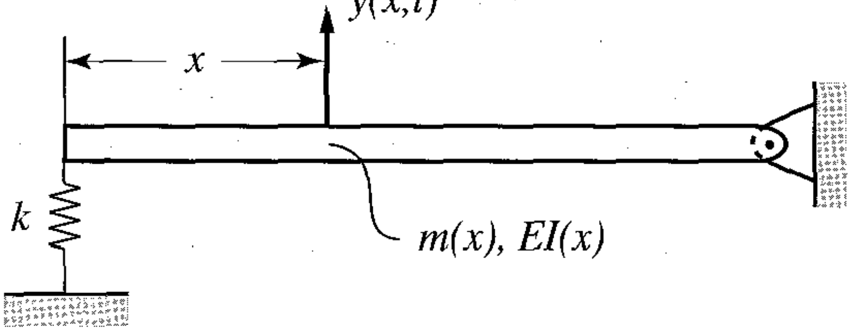
$$\int_0^L T(x)\frac{dY_r(x)}{dx}\frac{dY_s(x)}{dx}dx = \omega_r^2\delta_{rs},$$

ORTHOGONALITY OF MODES. EXPANSION THEOREM

To demonstrate the orthogonality relations for beams, we consider two distinct solutions of the eigenvalue problem:

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]=\omega_r^2 m(x)Y_r(x),\ 0<x<L$$

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_s(x)}{dx^2}\right]=\omega_s^2 m(x)Y_s(x),$$



$$Y(x)=0,\ EI(x)\frac{d^2Y(x)}{dx^2}=0,\ x=L$$

$$EI(x)\frac{d^2Y(x)}{dx^2}=0,\ -\frac{d}{dx}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=kY(x),\ x=0$$

Orthogonality relations for beams

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\omega_r^2\int_0^L m(x)Y_s(x)Y_r(x)dx$$

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\left\{Y_s(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\right\}\Big|_0^L$$

$$-\left[\frac{dY_s(x)}{dx}EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\Big|_0^L$$

$$+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx$$

$$=kY_s(0)Y_r(0)+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx$$

$$kY_s(0)Y_r(0)+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx=\omega_r^2\int_0^L m(x)Y_s(x)Y_r(x)dx$$

Orthogonality relations for beams

$$kY_s(0)Y_r(0)+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx=\omega_r^2\int_0^L m(x)Y_s(x)Y_r(x)dx$$

$$kY_r(0)Y_s(0)+\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx=\omega_s^2\int_0^L m(x)Y_r(x)Y_s(x)dx$$

$$\int_0^L m(x)Y_r(x)Y_s(x)=0,\ r,s=1,2,\ldots;\ \omega_r^2\neq\omega_s^2$$

$$\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx+kY_r(0)Y_s(0)=0,$$

Expansion Theorem:

Any function Y(x) representing a possible displacement of the system, with certain continuity, can be expanded in the absolutely and uniformly convergent series of the eigenfunctions:

$$Y(x) = \sum_{r=1}^{\infty} c_r Y_r(x)$$

$$c_r = \int_0^L m(x) Y_r(x) Y(x) dx, \; r = 1, 2, \ldots$$

The expansion theorem forms the basis for modal analysis, which permits the derivation of the response to both initial excitations and applied forces.

Distributed-Parameter Systems: Exact Solutions

- Relation between Discrete and Distributed Systems .
- Transverse Vibration of Strings
- Derivation of the String Vibration Problem by the Extended Hamilton Principle
- Bending Vibration of Beams
- Free Vibration: The Differential Eigenvalue Problem
- Orthogonality of Modes Expansion Theorem
- Systems with Lumped Masses at the Boundaries
- Eigenvalue Problem and Expansion Theorem for Problems with Lumped Masses at the Boundaries
- Rayleigh's Quotient . The Variational Approach to the Differential Eigenvalue Problem
- Response to Initial Excitations
- Response to External Excitations
- Systems with External Forces at Boundaries
- The Wave Equation
- Traveling Waves in Rods of Finite Length

Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 12)

By: H. Ahmadian

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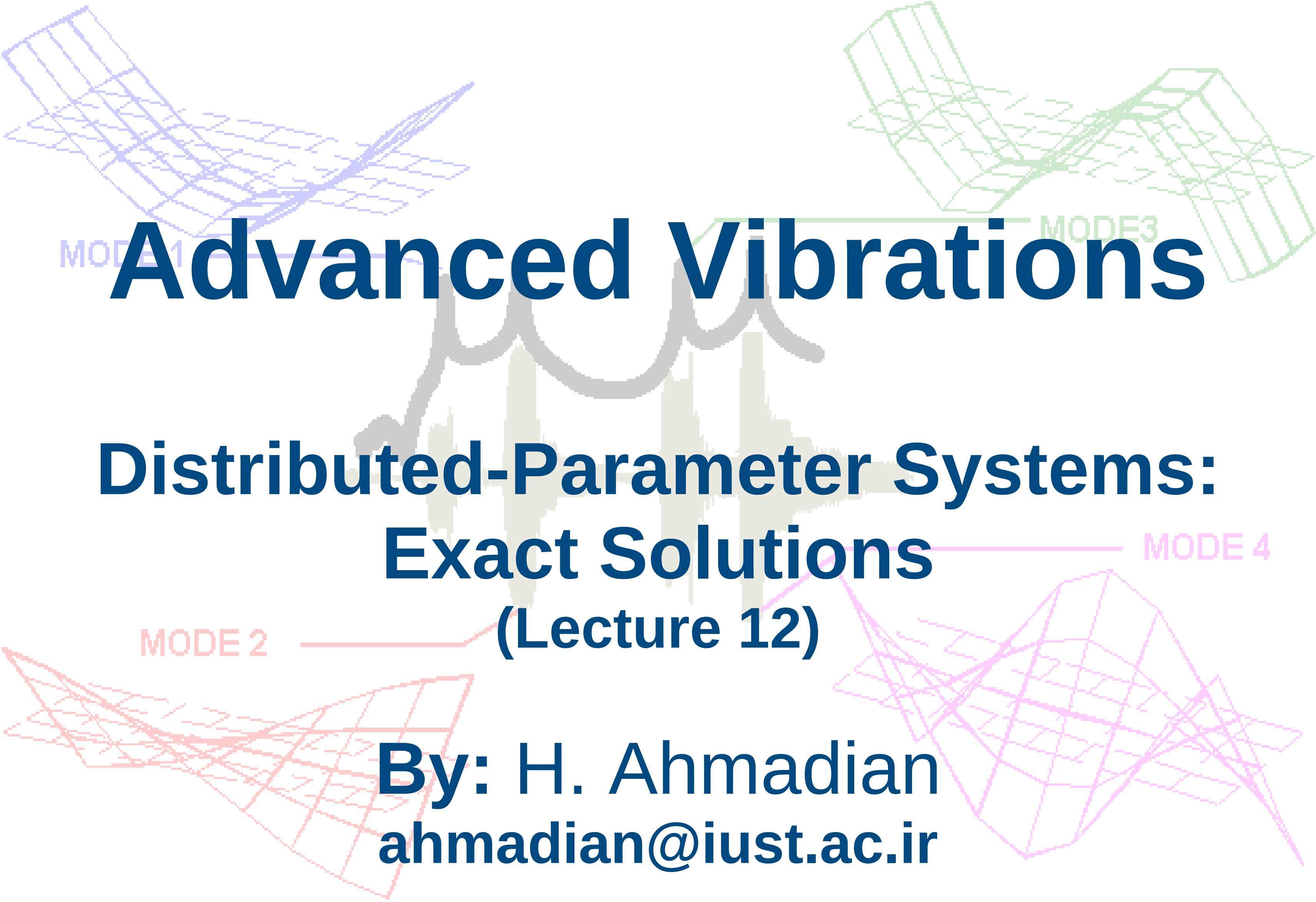

UMASS LOWELL MODAL ANALYSIS and CONTROLS LABORATORY - Pete Avitabile and Fabio Piergentili

School of Mechanical Engineering
Iran University of Science and Technology

Distributed-Parameter Systems: Exact Solutions

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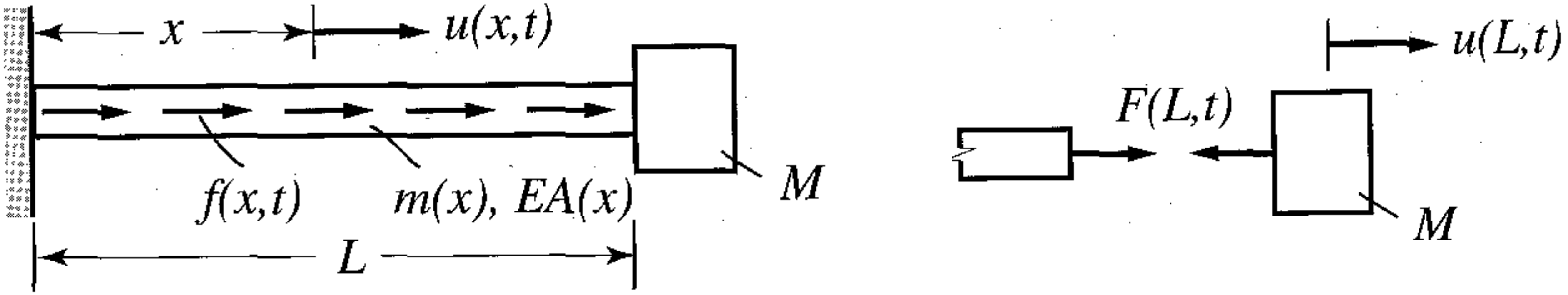
SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Rod with Tip Mass



$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]+f(x,t)=m(x)\frac{\partial^2 u(x,t)}{\partial t^2},\quad 0<x<L$$

$$u(0,t)=0 \qquad -EA(x)\frac{\partial u(x,t)}{\partial x}=M\frac{\partial^2 u(x,t)}{\partial t^2},\quad x=L$$

SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Rod with Tip Mass

By means of the extended Hamilton's principle:

$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W}_{nc})dt = 0,$$

$$\delta u(x,t) = 0,\ 0 \leq x \leq L,\ t = t_1, t_2$$

$$T(t) = \frac{1}{2}\int_0^L m(x)\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx + \frac{1}{2}M\left[\frac{\partial u(L,t)}{\partial t}\right]^2$$

$$V(t) = \frac{1}{2}\int_0^L EA(x)\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx$$

$$\overline{\delta W}_{nc} = \int_0^L f(x,t)\delta u(x,t)dx$$

SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Rod with Tip Mass

$$\int_{t_1}^{t_2} \delta T\,dt = \int_{t_1}^{t_2} \left[\int_0^L m(x) \frac{\partial u(x,t)}{\partial t} \delta \frac{\partial u(x,t)}{\partial t} dx + M \frac{\partial u(L,t)}{\partial t} \delta \frac{u(L,t)}{\delta t} \right] dt$$

$$= \int_{t_1}^{t_2} \left[\int_0^L m(x) \frac{\partial u(x,t)}{\partial t} \frac{\partial}{\partial t} \delta u(x,t) dx + M \frac{\partial u(L,t)}{\partial t} \frac{\partial}{\partial t} \delta u(L,t) \right] dt$$

$$= \int_0^L \left[m(x) \frac{\partial u(x,t)}{\partial t} \delta u(x,t) \Big|_{t_1}^{t_2} \right] dx - \int_0^L \left[\int_{t_1}^{t_2} m(x) \frac{\partial^2 u(x,t)}{\partial t^2} \delta u(x,t) dt \right] dx$$

$$+ M \frac{\partial u(L,t)}{\partial t} \delta u(L,t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} M \frac{\partial^2 u(L,t)}{\partial t^2} \delta u(L,t) dt$$

$$= - \int_{t_1}^{t_2} \left[\int_0^L m(x) \frac{\partial^2 u(x,t)}{\partial t^2} \delta u(x,t) dx + M \frac{\partial^2 u(L,t)}{\partial t^2} \delta u(L,t) \right] dt$$

SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Rod with Tip Mass

$$\delta V = \int_0^L EA(x)\frac{\partial u(x,t)}{\partial x}\delta\frac{\partial u(x,t)}{\partial x}dx = \int_0^L EA(x)\frac{\partial u(x,t)}{\partial x}\frac{\partial}{\partial x}\delta u(x,t)dx$$

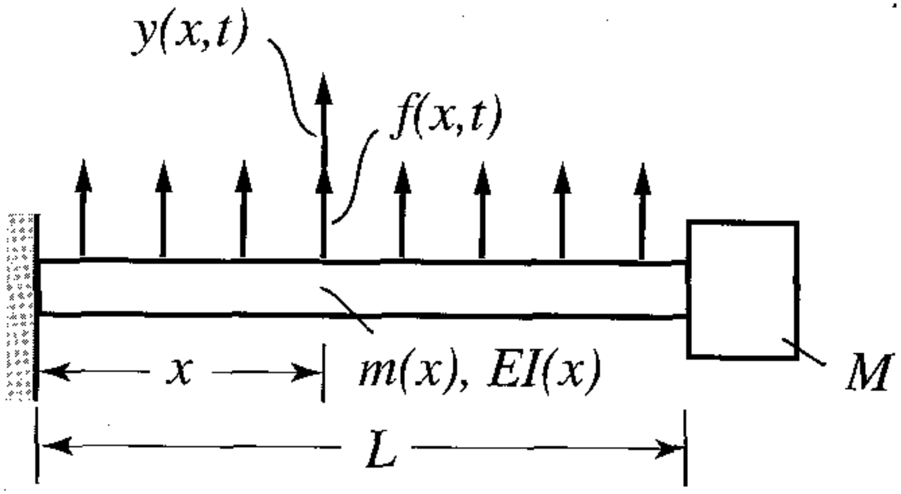
$$= EA(x)\frac{\partial u(x,t)}{\partial x}\delta u(x,t)\bigg|_0^L - \int_0^L \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]\delta u(x,t)dx$$

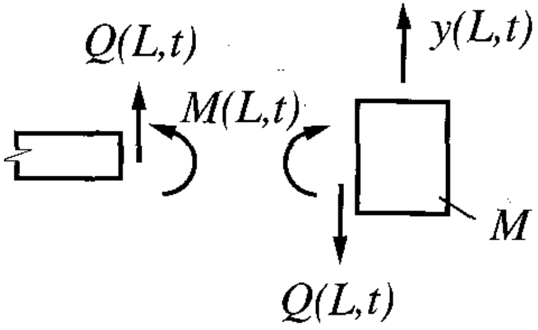
$$\int_{t_1}^{t_2}\Bigg[-\int_0^L\left\{m(x)\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] - f(x,t)\right\}\delta u(x,t)dx$$

$$-\left[EA(x)\frac{\partial u(x,t)}{\partial x} + M\frac{\partial^2 u(x,t)}{\partial t^2}\right]\delta u(x,t)\bigg|_{x=L}$$

$$+EA(x)\frac{\partial u(x,t)}{\partial x}\delta u(x,t)\bigg|_{x=0}\Bigg]dt = 0$$

SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Beam with Lumped Tip Mass





$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\quad 0<x<L$$

$$y(x,t)=0,\quad \frac{\partial y(x,t)}{\partial x}=0,\quad x=0$$

$$EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}=0,\qquad \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=M\frac{\partial^2 y(x,t)}{\partial t^2},\quad x=L$$

SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Beam with Tip Mass

By means of the extended Hamilton's principle:

$$T(t)=\frac{1}{2}\int_0^L m(x)\left[\frac{\partial y(x,t)}{\partial t}\right]^2 dx+\frac{1}{2}M\left[\frac{\partial y(L,t)}{\partial t}\right]^2$$

$$V(t)=\frac{1}{2}\int_0^L EI(x)\left[\frac{\partial^2 y(x,t)}{\partial x^2}\right]^2 dx$$

$$\int_{t_1}^{t_2}\delta T(t)dt=-\int_{t_1}^{t_2}\left[\int_0^L m(x)\frac{\partial^2 y(x,t)}{\partial t^2}\delta y(x,t)dx+M\frac{\partial^2 y(L,t)}{\partial t^2}\delta y(L,t)\right]dt$$

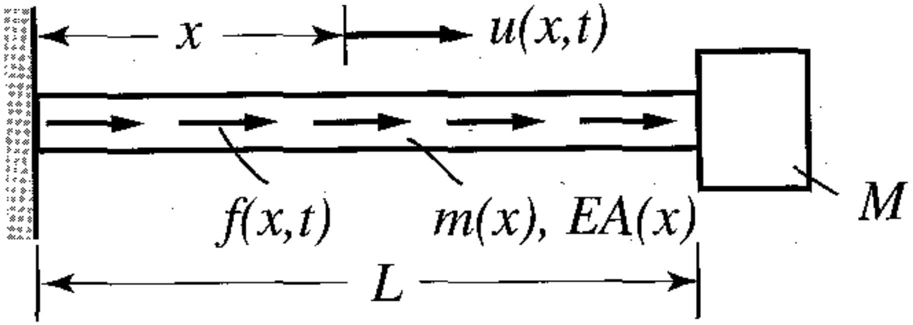
$$\delta V(t)=EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\delta\frac{\partial y(x,t)}{\partial x}\Big|_0^L-\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]\delta y(x,t)\Big|_0^L$$

$$+\int_0^L\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]\delta y(x,t)dx$$

SYSTEMS WITH LUMPED MASSES AT THE BOUNDARIES: Beam with Tip Mass

$$
\begin{aligned}
&\int_{t_1}^{t_2}\left\langle -\int_0^L \left\{ m(x)\frac{\partial^2 y(x,t)}{\partial t^2} - f(x,t) + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]\right\}\delta y(x,t)dx \right.\\
&\qquad - EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\delta\frac{\partial y(x,t)}{\partial x}\bigg|_{x=L} + EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\delta\frac{\partial y(x,t)}{\partial x}\bigg|_{x=0}\\
&\qquad + \left\{\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right] - M\frac{\partial^2 y(x,t)}{\partial t^2}\right\}\delta y(x,t)\bigg|_{x=L}\\
&\qquad \left. - \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]\delta y(x,t)\bigg|_{x=0}\right\rangle dt = 0
\end{aligned}
$$

EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES



$$u(0,t)=0$$

$$-EA(x)\frac{\partial u(x,t)}{\partial x}=M\frac{\partial^2 u(x,t)}{\partial t^2},\quad x=L$$

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]+f(x,t)=m(x)\frac{\partial^2 u(x,t)}{\partial t^2},\quad 0<x<L$$

$$u(x,t)=CU(x)\cos(\omega t-\phi)$$

$$-\frac{d}{dx}\left[EA(x)\frac{dU(x)}{dx}\right]=\omega^2 m(x)U(x),\quad 0<x<L$$

$$U(0)=0 \qquad EA(x)\frac{dU(x)}{dx}=\omega^2 MU(x),\quad x=L$$

EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES

The orthogonality of modes:

$$-\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]=\omega_r^2 m(x)U_r(x), \quad -\frac{d}{dx}\left[EA(x)\frac{dU_s(x)}{dx}\right]=\omega_s^2 m(x)U_s(x),$$

$$-\int_0^L U_s(x)\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]dx=\omega_r^2\int_0^L m(x)U_s(x)U_r(x)dx$$

$$-\int_0^L U_s(x)\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]dx$$

$$=-U_s(x)EA(x)\frac{dU_r(x)}{dx}\Big|_0^L+\int_0^L \frac{dU_s(x)}{dx}EA(x)\frac{dU_r(x)}{dx}dx$$

$$=-\omega_r^2 MU_s(L)U_r(L)+\int_0^L EA(x)\frac{dU_s(x)}{dx}\frac{dU_r(x)}{dx}dx$$

EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES

$$\int_0^L EA(x)\frac{dU_r(x)}{dx}\frac{dU_s(x)}{dx}dx = \omega_r^2\left[\int_0^L m(x)U_r(x)U_s(x)dx + MU_r(L)U_s(L)\right]$$

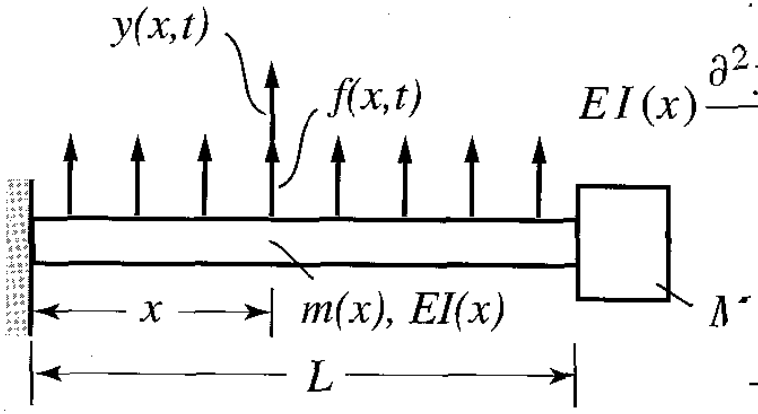
$$\int_0^L EA(x)\frac{dU_r(x)}{dx}\frac{dU_s(x)}{dx}dx = \omega_s^2\left[\int_0^L m(x)U_r(x)U_s(x)dx + MU_r(L)U_s(L)\right]$$

$$(\omega_r^2 - \omega_s^2)\left[\int_0^L m(x)U_r(x)U_s(x)dx + MU_r(L)U_s(L)\right] = 0$$

$$\int_0^L m(x)U_r(x)U_s(x)dx + MU_r(L)U_s(L) = \delta_{rs},$$

$$\int_0^L EA(x)\frac{dU_r(x)}{dx}\frac{dU_s(x)}{dx}dx = \omega_r^2\delta_{rs},\ \ r,s = 1,2,\ldots$$

EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES



$$EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}=0,\quad \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=M\frac{\partial^2 y(x,t)}{\partial t^2},\quad x=L$$

$$y(x,t)=0,\quad \frac{\partial y(x,t)}{\partial x}=0,\quad x=0$$

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},$$

$$y(x,t)=CY(x)\cos(\omega t-\phi)$$

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=\omega^2 m(x)Y(x),$$

$$Y(x)=0,\quad \frac{dY(x)}{dx}=0,\quad x=0$$

$$EI(x)\frac{d^2Y(x)}{dx^2}=0,\quad -\frac{d}{dx}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]=\omega^2 MY(x),\quad x=L$$

EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\omega_r^2\int_0^L m(x)Y_s(x)Y_r(x)dx$$

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx$$

$$=\left\{Y_s(x)\frac{d}{dx}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\right\}\Bigg|_0^L-\left[\frac{dY_s(x)}{dx}EI(x)\frac{d^2Y_r(x)}{dx^2}\right]\Bigg|_0^L$$

$$+\int_0^L\frac{d^2Y_s(x)}{dx^2}EI(x)\frac{d^2Y_r(x)}{dx^2}dx$$

$$=-\omega_r^2MY_s(L)Y_r(L)+\int_0^L EI(x)\frac{d^2Y_s(x)}{dx^2}\frac{d^2Y_r(x)}{dx^2}dx$$

EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES

$$\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx = \omega_r^2\left[\int_0^L m(x)Y_r(x)Y_s(x)dx + MY_r(L)Y_s(L)\right]$$

$$\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx = \omega_s^2\left[\int_0^L m(x)Y_r(x)Y_s(x)dx + MY_r(L)Y_s(L)\right]$$

$$(\omega_r^2 - \omega_s^2)\left[\int_0^L m(x)Y_r(x)Y_3(x)dx + MY_r(L)Y_s(L)\right] = 0$$

$$\int_0^L m(x)Y_r(x)Y_s(x)dx + MY_r(L)Y_s(L) = \delta_{rs},$$

$$\int_0^L EI(x)\frac{d^2Y_r(x)}{dx^2}\frac{d^2Y_s(x)}{dx^2}dx = \omega_r^2\delta_{rs},\quad r, s = 1, 2, \ldots$$

Example 8.6. The eigenvalue problem for a uniform circular shaft in torsion

$$-\frac{d}{dx}\left[GJ(x)\frac{d\Theta(x)}{dx}\right]=\omega^2 I(x)\Theta(x),\ 0<x<L$$

$$\Theta(0)=0 \qquad GJ(x)\frac{d\theta(x)}{dx}\bigg|_{x=L}=\omega^2 I_D\Theta(L)$$

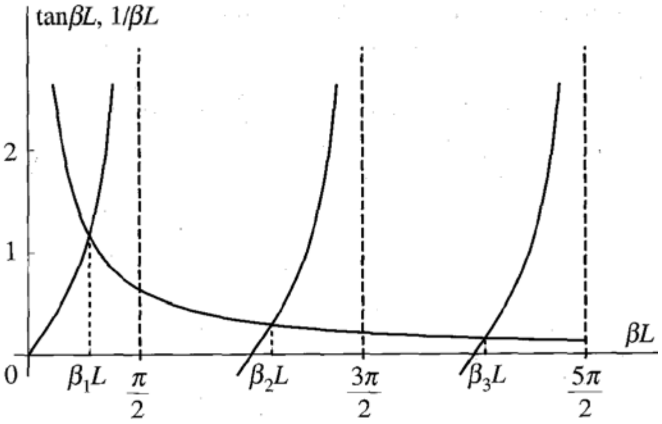
$$\frac{d^2\Theta(x)}{dx^2}+\beta^2\Theta(x)=0,\ 0<x<L,\ \beta^2=\frac{\omega^2 I}{GJ}$$

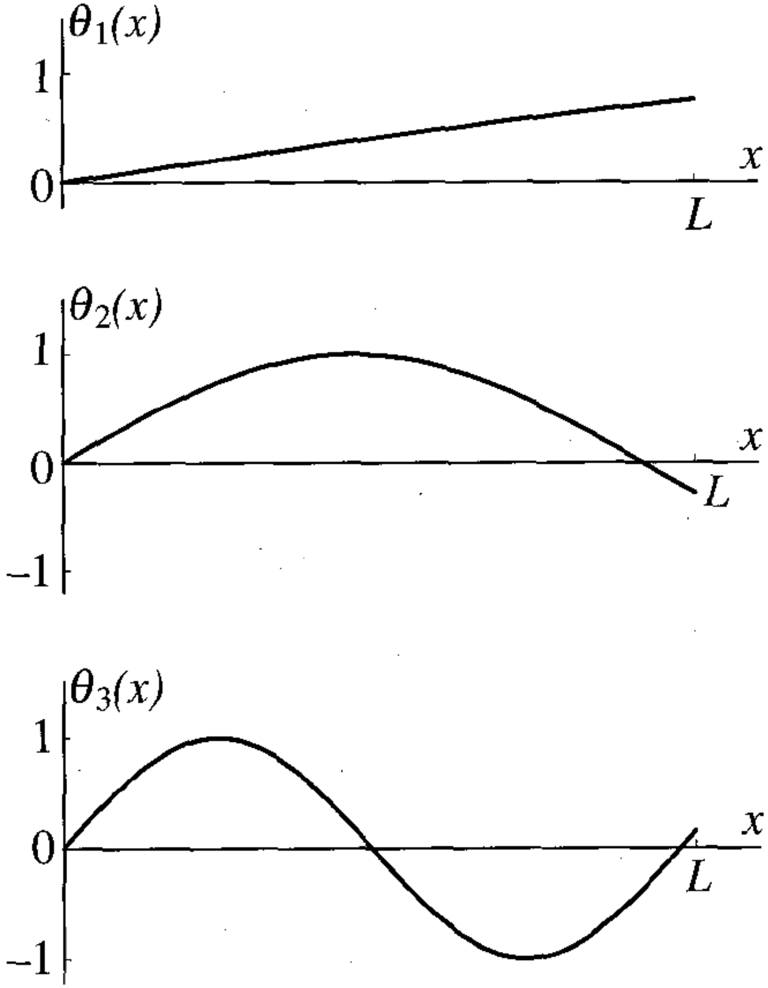
$$\Theta(x)=A\sin\beta x+B\cos\beta x$$

$$\Theta(0)=0 \Longrightarrow B=0.$$

$$\frac{d\Theta(x)}{dx}\bigg|_{x=L}=\frac{\beta^2 I_D}{I}\Theta(L) \Longrightarrow \tan\beta L=\frac{1}{\beta L}$$

Example 8.6. The eigenvalue problem for a uniform circular shaft in torsion





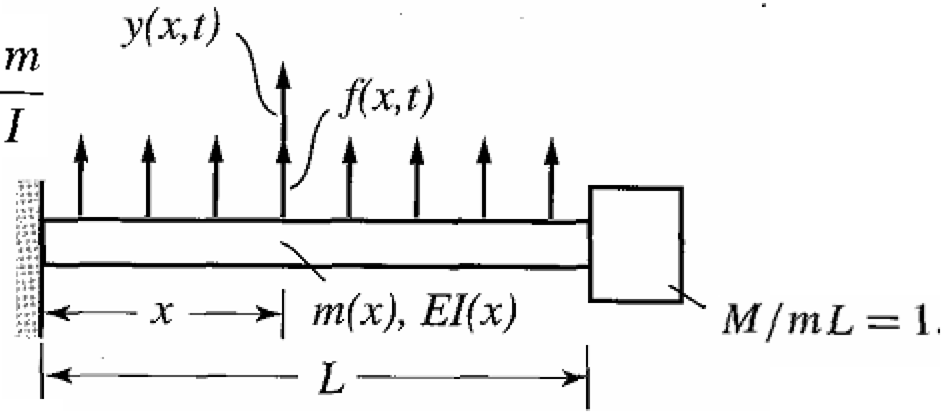
$$\omega_1 = 0.8903\sqrt{\frac{GJ}{IL^2}}$$

$$\omega_2 = 3.4256\sqrt{\frac{GJ}{IL^2}}$$

$$\omega_3 = 6.4373\sqrt{\frac{GJ}{IL^2}}$$

Example 8.7. The eigenvalue problem for a uniform cantilever beam with tip mass

$$\frac{d^4Y(x)}{dx^4} - \beta^4 Y(x) = 0,\ 0 < x < L,\ \beta^4 = \frac{\omega^2 m}{EI}$$



$$Y(x) = 0,\ \frac{dY(x)}{dx} = 0,\ x = 0$$

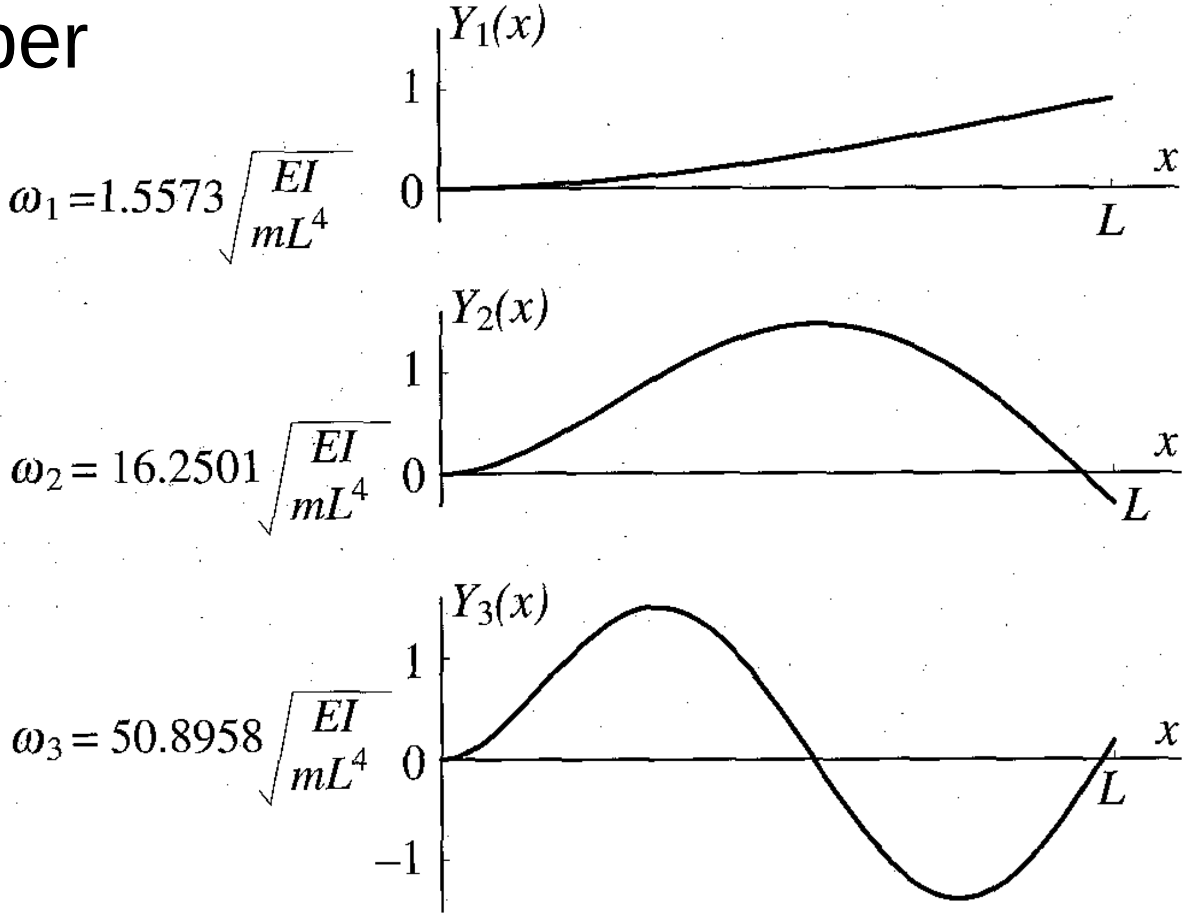
$$\frac{d^2Y(x)}{dx^2} = 0,\ \frac{d^3Y(x)}{dx^3} + \frac{M}{m}\beta^4 Y(x) = 0,\ x = L$$

$$-(1 + \cos\beta L \cosh\beta L) + \beta L(\sin\beta L \cosh\beta L - \sinh\beta L \cos\beta L) = 0$$

$$Y_r(x) = A_r\left[\sin\beta_r x - \sinh\beta_r x - \frac{\sin\beta_r L + \sinh\beta_r L}{\cos\beta_r L + \cosh\beta_r L}(\cos\beta_r x - \cosh\beta_r x)\right]$$

Example 8.7. The eigenvalue problem for a uniform cantilever beam with tip mass

As the mode number increases, the end acts more as a pinned end



EIGENVALUE PROBLEM AND EXPANSION THEOREM FOR PROBLEMS WITH LUMPED MASSES AT THE BOUNDARIES

Any function U(x) representing a possible displacement of the continuous model, which implies that U(x) satisfies boundary conditions and is such that its derivatives up to the order appeared in the model is a continuous function, can be expanded in the absolutely and uniformly convergent series of the eigenfunctions:

$$U(x) = \sum^{\infty} c_r U_r(x)$$

$$c_r = \int_0^L m(x) U_r(x) U(x) dx + M U_r(L) U(L), \quad r = 1, 2, \ldots$$

Distributed-Parameter Systems: Exact Solutions

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Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 13)

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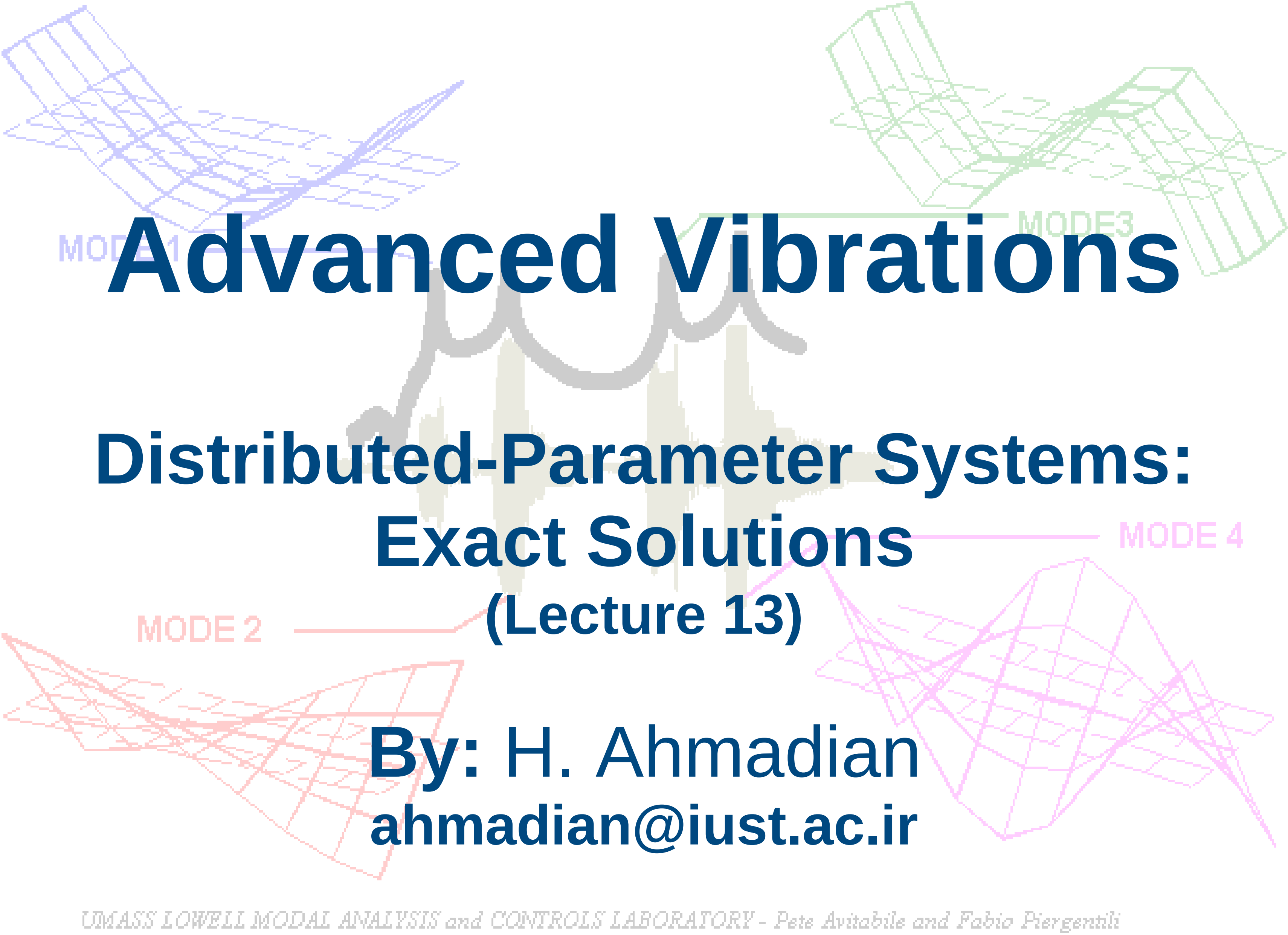

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RAYLEIGH'S QUOTIENT. VARIATIONAL APPROACH TO THE DIFFERENTIAL EIGENVALUE PROBLEM

- Cases in which the differential eigenvalue problem admits a closed-form solution are very rare :
  - Uniformly distributed parameters and
  - Simple boundary conditions.
- For the most part, one must be content with approximate solutions,
  - Rayleigh's quotient plays a pivotal role.

The *strong form* of the eigenvalue problem

- A rod in axial vibration fixed at ***x=0*** and with a spring of stiffness ***k*** at ***x=L***.

$$-\frac{d}{dx}\left[EA(x)\frac{dU(x)}{dx}\right]=\lambda m(x)U(x),\ 0<x<L;\ \lambda=\omega^2$$

$$U(0)=0,\ -EA(x)\frac{dU(x)}{dx}\bigg|_{x=L}=kU(L)$$

- An exact solution of the eigenvalue problem in the strong form is beyond reach,
  - The mass and stiffness parameters depend on the spatial variable ***x*** .

The differential eigenvalue problem in a *weak form*

$$-\int_0^L V(x)\frac{d}{dx}\left[EA(x)\frac{dU(x)}{dx}\right]dx = \lambda\int_0^L m(x)V(x)U(x)dx$$

Test function

- The solution of the differential eigenvalue problem is in a weighted average sense
- The test function *V(x)* plays the role of a weighting function.
- The test function *V(x)* satisfies the geometric boundary conditions and certain continuity requirments.

The differential eigenvalue problem in a *weak form*

$$-\int_0^L V(x)\frac{d}{dx}\left[EA(x)\frac{dU(x)}{dx}\right]dx=\lambda\int_0^L m(x)V(x)U(x)dx$$

Symmetrizing the left side

$$\int_0^L EA(x)\frac{dV(x)}{dx}\frac{dU(x)}{dx}dx+kV(L)U(L)=\lambda\int_0^L m(x)V(x)U(x)dx$$

The differential eigenvalue problem in a *weak form: Rayleigh's quotient*

- We consider the case in which the test function is equal to the trial function:

$$R(U) = \lambda = \omega^2 = \frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx + kU^2(L)}{\int_0^L m(x)U^2(x)dx}$$

- The value of R depends on the trial function
- How the value of R behaves as *U(x)* changes?

Properties of Rayleigh's quotient

$$U(x)=\sum_{i=1}^{\infty}c_iU_i(x)$$

$$\int_0^L m(x)U_i(x)U_j(x)dx=\delta_{ij},\ i,j=1,2,\dots$$

$$\int_0^L EA(x)\frac{dU_i(x)}{dx}\frac{dU_j(x)}{dx}dx+kU_i(L)U_j(L)=\lambda_i\delta_{ij}$$

$$R(c_1,c_2,\dots)=\lambda=\omega^2$$

$$=\frac{\int_0^L EA(x)\sum_{i=1}^{\infty}c_i\frac{dU_i(x)}{dx}\sum_{j=1}^{\infty}c_j\frac{dU_j(x)}{dx}dx+k\sum_{i=1}^{\infty}c_iU_i(L)\sum_{j=1}^{\infty}c_jU_j(L)}{\int_0^L m(x)\sum_{i=1}^{\infty}c_iU_i(x)\sum_{j=1}^{\infty}c_jU_j(x)dx}$$

Properties of Rayleigh's quotient

$$R(c_1, c_2, \ldots) = \lambda = \omega^2$$

$$= \frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_i c_j \left[\int_0^L EA(x)\frac{dU_i(x)}{dx}\frac{dU_j(x)}{dx}dx + kU_i(L)U_j(L)\right]}{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_i c_j \int_0^L m(x)U_i(x)U_j(x)dx}$$

$$= \frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_i c_j \lambda_i \delta_{ij}}{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_i c_j \delta_{ij}} = \frac{\sum_{i=1}^{\infty} c_i^2 \lambda_i}{\sum_{i=1}^{\infty} c_i^2}$$

Properties of Rayleigh's quotient

$$c_i = \epsilon_i c_r, \quad i = 1, 2, \ldots, r-1, r+1, \ldots$$

$$R = \frac{c_r^2 \lambda_r + \sum\limits_{\substack{i=1\\ i\neq r}}^{\infty} c_i^2 \lambda_i}{c_r^2 + \sum\limits_{\substack{i=1\\ i\neq r}}^{\infty} c_i^2} = \frac{\lambda_r + \sum\limits_{\substack{i=1\\ i\neq r}}^{\infty} \epsilon_i^2 \lambda_i}{1 + \sum\limits_{\substack{i=1\\ 1\neq r}}^{\infty} \epsilon_i^2}$$

$$\cong \left(\lambda_r + \sum_{\substack{i=1\\ i\neq r}}^{\infty} \epsilon_i^2 \lambda_i\right)\left(1 - \sum_{\substack{i=1\\ i\neq r}}^{\infty} \epsilon_i^2\right) \cong \lambda_r + \sum_{i=1}^{\infty} (\lambda_i - \lambda_r)\epsilon_i^2$$

Properties of Rayleigh's quotient

- The trial function *U(x)* differs from the rth eigenfunction $U_r(x)$ by a small quantity of first order ϵ, or $U(x) = U_r(x) + O(\epsilon)$, *and* Rayleigh's quotient differs from the rth eigenvalue by a small quantity of second order in ϵ, or $R = \lambda_r + O(\epsilon^2)$.
- *Rayleigh 's quotient has a stationary value at an eigenfinction* $U_r(x)$*, where the stationary value is the associated eigenvalue*.

Properties of Rayleigh's quotient

$r = 1,$

$$R \cong \lambda_1 + \sum_{i=2}^{\infty} (\lambda_i - \lambda_1)\epsilon_i^2$$

$$R \geq \lambda_1$$

$$\lambda_1 = \omega_1^2 = \min R(U) = R(U_1)$$

Rayleigh's quotient

A fixed-tip mass rod:

$$R(U)=\lambda=\omega^2=\frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L m(x)U^2(x)dx+MU^2(L)}$$

A pinned-spring supported beam in bending:

$$R(Y)=\lambda=\omega^2=\frac{\int_0^L EI(x)\left[\frac{d^2Y(x)}{dx^2}\right]^2 dx+kY^2(0)}{\int_0^L m(x)Y^2(x)dx}$$

Rayleigh's quotient

- Rayleigh's quotient for all systems have one thing in common:
  - the numerator is a measure of the potential energy
  - and the denominator a measure of the kinetic energy.

$$R = \lambda = \omega^2 = \frac{V_{\max}}{T_{\mathrm{ref}}}$$

A fixed-spring supported rod in axial vibration

$$V(t) = \frac{1}{2}\int_0^L EA(x)\left[\frac{\partial u(x,t)}{\partial x}\right]^2 dx + \frac{1}{2}ku^2(L,t)$$

$$T(t) = \frac{1}{2}\int_0^L m(x)\left[\frac{\partial u(x,t)}{\partial t}\right]^2 dx$$

A fixed-spring supported rod in axial vibration

$$u(x,t) = U(x)\cos(\omega t - \phi)$$

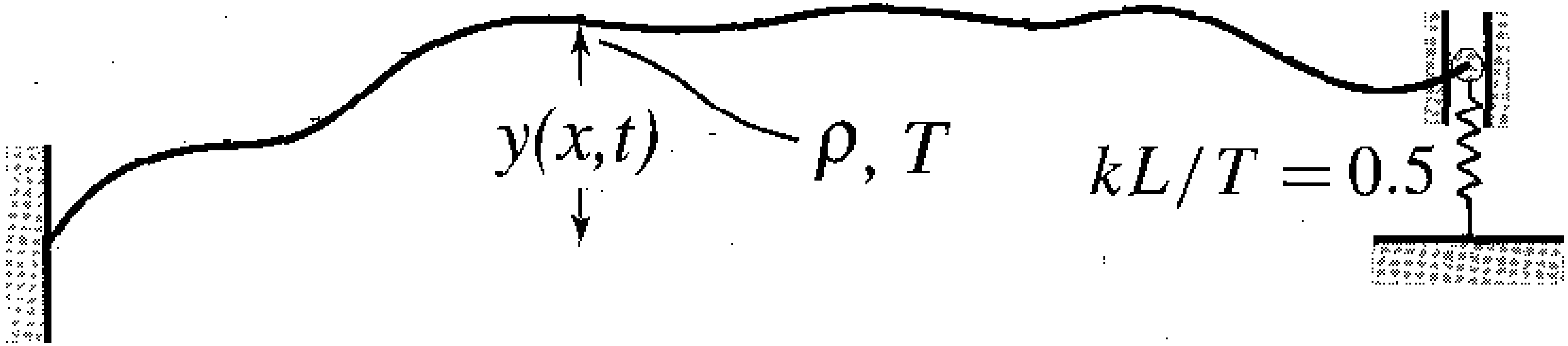
$$V(t) = \frac{1}{2}\left\{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx + kU^2(L)\right\}\cos^2(\omega t - \phi)$$

$$= V_{\max}\cos^2(\omega t - \phi)$$

$$T(t) = \frac{\omega^2}{2}\left[\int_0^L m(x)U^2(x)dx\right]\sin^2(\omega t - \phi) = \omega^2 T_{\text{ref}}\sin^2(\omega t - \phi)$$

$$R = \lambda = \omega^2 = \frac{V_{\max}}{T_{\text{ref}}}$$

Example 8.8. Estimation of the lowest eigenvalue by means of Rayleigh's principle



$$R = \omega^2 = \frac{T\int_0^L \left[\frac{dY(x)}{dx}\right]^2 dx + kY^2(L)}{\rho\int_0^L Y^2(x)dx}$$

Example 8.8: a) The static displacement curve as a trial function

$$T\frac{d^2Y(x)}{dx^2}=\rho g,\ 0<x<L$$

$$Y(x)=c_1x+c_2+\frac{1}{2}\frac{\rho g}{T}x^2$$

$$Y(0)=0,\ T\frac{dY(x)}{dx}+kY(x)=0,\ \ x=L$$

$$c_1=-\frac{\rho g}{T}\frac{1+kL/2T}{1+kL/T},\ c_2=0$$

Example 8.8: a) The static displacement curve as a trial function

$$Y(x)=-\frac{5}{6}\frac{\rho g L}{T}x+\frac{1}{2}\frac{\rho g}{T}x^2=\frac{\rho g L^2}{T}\left[-\frac{5}{6}\frac{x}{L}+\frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

$$\omega^2=\frac{T\left(\frac{\rho g L}{T}\right)^2\int_0^L-\left(\frac{5}{6}+\frac{x}{L}\right)^2dx+k\left(\frac{\rho g L^2}{T}\right)^2\left(-\frac{5}{6}+\frac{1}{2}\right)^2}{\rho\left(\frac{\rho g L^2}{T}\right)^2\int_0^L\left[-\frac{5}{6}\frac{x}{L}+\frac{1}{2}\left(\frac{x}{L}\right)^2\right]^2dx}$$

$$=\frac{\frac{T}{L^2}\frac{7}{36}+\frac{k}{L}\frac{1}{9}}{\rho\frac{79}{1080}}=3.4177\frac{T}{\rho L^2}$$

$$\beta^2=\omega^2\rho/T \qquad \beta L=\sqrt{3.4177}=1.8487$$

$$e=\frac{\beta L-\beta_1 L}{\beta_1 L}=\frac{1.8487-1.8366}{1.8366}=0.0066=0.66\%$$

Example 8.8: b)The lowest eigenfunction of a fixed-free string as a trial function

$$Y(x) = \sin\frac{\pi x}{2L}$$

$$\omega^2 = \frac{T\left(\frac{\pi}{2L}\right)^2 \int_0^L \cos^2\frac{\pi x}{2L}dx + k}{\rho\int_0^L \sin^2\frac{\pi x}{2L}dx} = \frac{T\left(\frac{\pi}{2L}\right)^2 \frac{L}{2} + k}{\rho\frac{L}{2}}$$

$$= \left[\left(\frac{\pi}{2}\right)^2 + 1\right]\frac{T}{\rho L^2} = 3.4674\frac{T}{\rho L^2}$$

$$\beta L = \sqrt{3.4674} = 1.8621$$

$$e = \frac{\beta L - \beta_1 L}{\beta_1 L} = \frac{1.8621 - 1.8366}{1.8366} = 0.0139 = 1.39\%$$

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Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 14)

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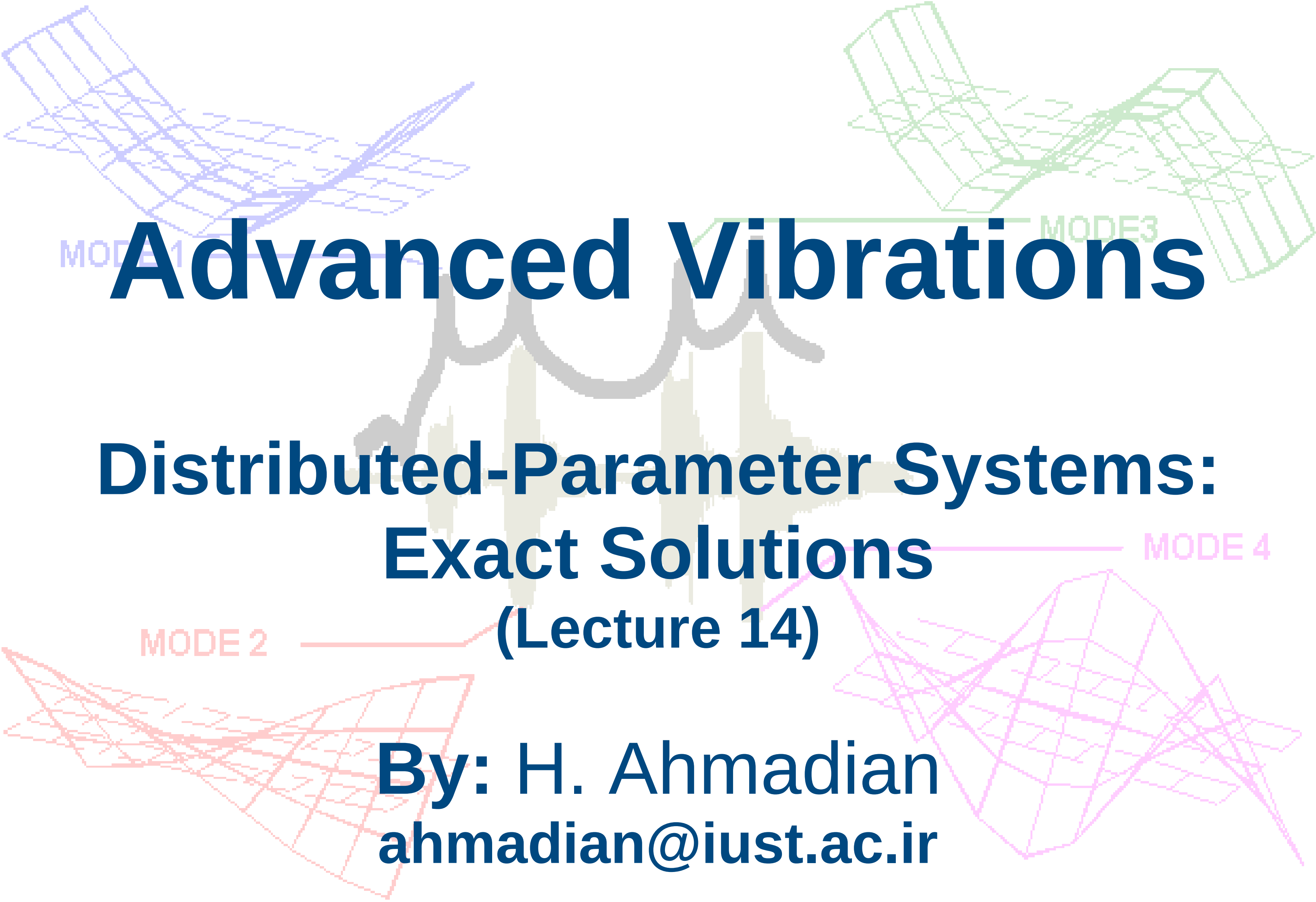

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RESPONSE TO INITIAL EXCITATIONS

- Various distributed-parameter systems exhibit similar vibrational characteristics, although their mathematical description tends to differ in appearance.
- Consider the transverse displacement *y(x,t)* of a string in free vibration

$$\frac{\partial}{\partial x}\left[T(x)\frac{\partial y(x,t)}{\partial x}\right]=\rho(x)\frac{\partial^2 y(x,t)}{\partial t^2},\quad 0<x<L$$

caused by initial excitations in the form of

$$y(x,0)=y_0(x),\quad \left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0}=v_0(x)$$

RESPONSE TO INITIAL EXCITATIONS

$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$$

the normal modes

$$\sum_{r=1}^{\infty} \frac{d}{dx}\left[T(x)\frac{dY_r(x)}{dx}\right]\eta_r(t) = \sum_{r=1}^{\infty} \rho(x)Y_r(x)\frac{d^2\eta_r(t)}{dt^2},$$

$$\sum_{r=1}^{\infty}\left\{\int_0^L Y_s(x)\frac{d}{dx}\left[T(x)\frac{dY_r(x)}{dx}\right]dx\right\}\eta_r(t) = \sum_{r=1}^{\infty}\left[\int_0^L \rho(x)Y_s(x)Y_r(x)dx\right]\frac{d^2\eta_r(t)}{dt^2}$$

$$\ddot{\eta}_r(t) + \omega_r^2\eta_r(t) = 0, \ r = 1, 2, \ldots$$

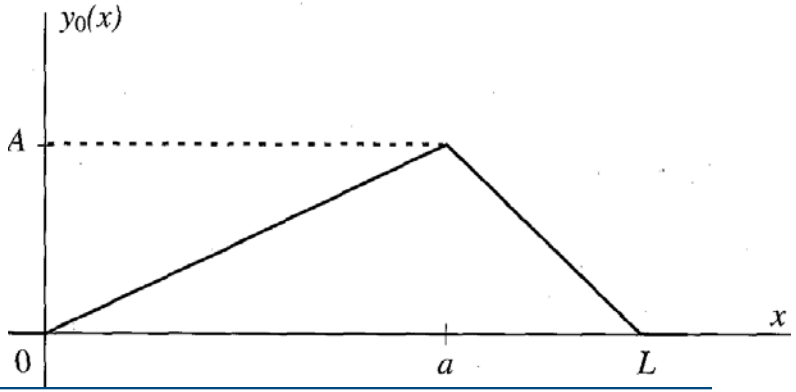
$$\eta_r(t) = \eta_r(0)\cos\omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r}\sin\omega_r t,$$

$$\eta_r(0) = \int_0^L \rho(x)Y_r(x)y_0(x)dx, \qquad \dot{\eta}_r(0) = \int_0^L \rho(x)Y_r(x)v_0(x)dx,$$

Example:

Response of a uniform string to the initial displacement $y_0(x)$and zero initial velocity.

$$y_0(x) = \begin{cases} \dfrac{Ax}{a}, \; 0 < x < a \\ \dfrac{A}{L-a}(L-x), \; a < x < L \end{cases}$$



$$y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$$

$$\omega_r = r\pi\sqrt{\frac{T}{\rho L^2}}, \; Y_r(x) = \sqrt{\frac{2}{\rho L}}\sin\frac{r\pi x}{L}, \; r = 1, 2, \ldots \; \int_0^L \rho(x) Y_r^2(x) dx = 1$$

Example:

$$\eta_r(t) = \eta_r(0)\cos\omega_r t, \quad r = 1, 2, \ldots$$

$$\eta_r(0) = \int_0^L \rho Y_r(x) y_0(x) dx = A\sqrt{2\rho L}\frac{L^2}{r^2\pi^2 a(L-a)}\sin\frac{r\pi a}{L},$$

$$y(x,t) = \frac{2AL^2}{\pi^2 a(L-a)}\sum_{r=1}^{\infty}\frac{1}{r^2}\sin\frac{r\pi a}{L}\sin\frac{r\pi x}{L}\cos r\pi\sqrt{\frac{T}{\rho L^2}}t$$

$$a = L/2$$

$$y(x,t) = \frac{2AL^2}{\pi^2 a(L-a)}\sum_{r=1,3,\ldots}^{\infty}\frac{(-1)^{(r-1)/2}}{r^2}\sin\frac{r\pi x}{L}\cos r\pi\sqrt{\frac{T}{\rho L^2}}t$$

RESPONSE TO INITIAL EXCITATIONS: Beams in Bending Vibration

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\ 0<x<L$$

$$-\sum_{r=1}^{\infty}\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 Y_r(x)}{dx^2}\right]\eta_r(t)=\sum_{r=1}^{\infty}m(x)Y_r(x)\frac{d^2\eta_r(t)}{dt^2},$$

$$-\sum_{r=1}^{\infty}\left\{\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2 Y_r(x)}{dx^2}\right]dx\right\}\eta_r(t)$$

$$=\sum_{r=1}^{\infty}\left[\int_0^L m(x)Y_s(x)Y_r(x)dx\right]\frac{d^2\eta_r(t)}{dt^2}$$

RESPONSE TO INITIAL EXCITATIONS: Beams in Bending Vibration

To demonstrate that every one of the natural modes can be excited independently of the other modes we select the initials as:

$$y_0(x) = AY_p(x)$$

$$\eta_r(0) = A\int_0^L \rho(x)Y_r(x)Y_p(x)dx = \begin{cases} A \text{ for } r = p \\ 0 \text{ for } r = 1, 2, \ldots, p-1, p+1, \ldots \end{cases}$$

$$\eta_r(t) = \begin{cases} A\cos\omega_r t \text{ for } r = p \\ 0 \text{ for } r = 1, 2, \ldots, p-1, p+1, \ldots \end{cases}$$

$$y(x,t) = AY_p(x)\cos\omega_p t$$

RESPONSE TO INITIAL EXCITATIONS:

Response of systems with tip masses

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] = m(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \quad 0 < x < L$$

Boundary conditions

$$u(0,t) = 0$$

$$-EA(x)\frac{\partial u(x,t)}{\partial x} = M\frac{\partial^2 u(x,t)}{\partial t^2}, \quad x = L$$

Initial conditions

$$u(x,0) = u_0(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = v_0(x)$$

RESPONSE TO INITIAL EXCITATIONS:

Response of systems with tip masses

$$u(x,t)=\sum_{r=1}^{\infty}U_r(x)\eta_r(t)$$

$$\sum_{r=1}^{\infty}\left\{\int_0^L U_s(x)\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]dx\right\}\eta_r(t)=\sum_{r=1}^{\infty}\left[\int_0^L m(x)U_s(x)U_r(x)dx\right]\ddot{\eta}_r(t),$$

$$\int_0^L m(x)U_s(x)U_s(x)dx=\delta_{rs}-MU_r(L)U_s(L),$$

$$\int_0^L U_s(x)\frac{d}{dx}\left[EA(x)\frac{dU_r(x)}{dx}\right]dx=\left[U_s(x)EA(x)\frac{dU_r(x)}{dx}\right]\Bigg|_{x=L}-\omega_r^2\delta_{rs},$$

Observing from boundary condition →

$$\sum_{r=1}^{\infty}\left[MU_r(x)\ddot{\eta}_r(t)+EA(x)\frac{dU_r(x)}{dx}\eta_r(t)\right]\Bigg|_{x=L}$$

$$=\left[M\frac{\partial^2u(x,t)}{\partial t^2}+EA(x)\frac{\partial u(x,t)}{\partial x}\right]\Bigg|_{x=L}=0$$

RESPONSE TO INITIAL EXCITATIONS:

Response of systems with tip masses

$$\ddot{\eta}_s(t) + \omega_s^2 \eta_s(t) = 0, \quad s = 1, 2, \ldots$$

$$\eta_s(t) = \eta_s(0)\cos\omega_s t + \frac{\dot{\eta}_s(0)}{\omega_s}\sin\omega_s t,$$

$$u(x,0) = \sum_{s=1}^{\infty} U_s(x)\eta_s(0) = u_0(x)$$

$$\eta_s(0) = \int_0^L m(x)U_s(x)u_0(x)dx + MU_s(L)u_0(L),$$

Similarly,

$$\dot{\eta}_s(0) = \int_0^L m(x)U_s(x)v_0(x)dx + MU_s(L)v_0(L),$$

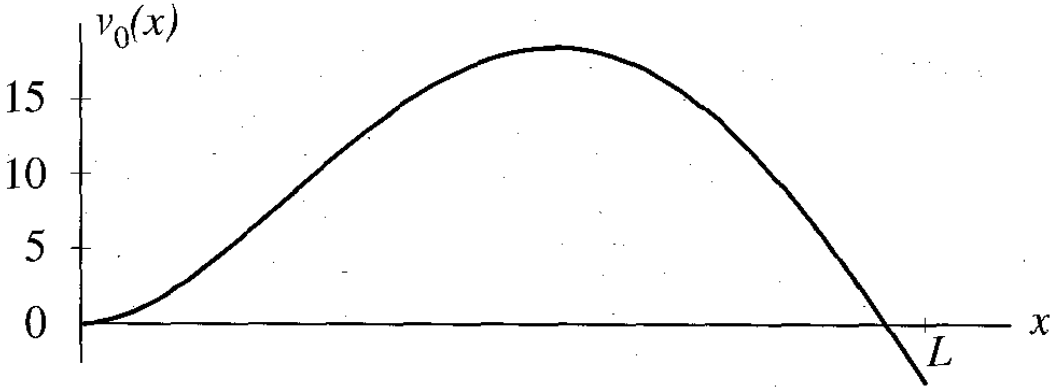
Example:

Response of a cantilever beam with a lumped mass at the end to the initial velocity:

$$-EI\frac{\partial^4 y(x,t)}{\partial x^4} = m\frac{\partial^2 y(x,t)}{\partial t^2},\ 0 < x < L$$

$$y(x,t) = 0,\ \frac{\partial y(x,t)}{\partial x} = 0,\ x = 0 \qquad EI\frac{\partial^2 y(x,t)}{\partial x^2} = 0,\ EI\frac{\partial^3 y(x,t)}{\partial x^3} = M\frac{\partial^2 y(x,t)}{\partial t^2},\ x = L$$

$$v_0(x) = 13.72\left(\frac{x}{L}\right)^2 - 23.22\left(\frac{x}{L}\right)^3 + 9.26\left(\frac{x}{L}\right)^4$$



Example:

$$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)$$

$$m\int_0^L Y_r(x)Y_s(x)dx+MY_r(L)Y_s(L)=\delta_{rs},$$

$$EI\left\{\int_0^L Y_s(x)\frac{d^4Y_r(x)}{dx^4}dx-\left[Y_s(x)\frac{d^3Y_r(x)}{dx^3}\right]\Big|_{x=L}\right\}=\omega_r^2\delta_{rs},$$

$$\ddot{\eta}_s(t)+\omega_s^2\eta_s(t)-\sum_{r=1}^{\infty}\left\{Y_s(x)\left[MY_r(x)\ddot{\eta}_r(t)-EI\frac{d^3Y_r(x)}{dx^3}\eta_r(t)\right]\right\}\Big|_{x=L}=0,$$

$$\ddot{\eta}_c+\omega_s^2\eta_s(t)=0,\ s=1,2\ldots\quad \eta_s(t)=\frac{\dot{\eta}_s(0)}{\omega_s}\sin\omega_s t,$$

Example:

$$\dot{\eta}_s(0) = m\int_0^L Y_s(x)v_0(x)dx + MY_s(L)v_0(L)$$

$$= m\int_0^L Y_s(x)\left[13.72\left(\frac{x}{L}\right)^2 - 23.22\left(\frac{x}{L}\right)^3 + 9.26\left(\frac{x}{L}\right)^4\right]dx - 0.24MY_s(L),$$

$$M = mL,$$

$$y(x,t) = \sum_{r=1}^{\infty} C_r\left[\sin\beta_r x - \sinh\beta_r x - \frac{\sin\beta_r L + \sinh\beta_r L}{\cos\beta_r L + \cosh\beta_r L}(\cos\beta_r x - \cosh\beta_r x)\right]\sin\omega_r t$$

$$C_1 = -0.0404,\ C_2 = 0.7761,\ C_3 = -0.0003,$$

Because initial velocity resembles the 2nd mode

RESPONSE TO EXTERNAL EXCITATIONS

- The various types of distributed-parameter systems differ more in appearance than in vibrational characteristics.
- We consider the response of a beam in bending supported by a spring of stiffness ***k*** at *x=0* and pinned at *x=L*.

$$-\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+f(x,t)=m(x)\frac{\partial^2 y(x,t)}{\partial t^2},\quad 0<x<L$$

$$EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}=0,\ \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]+ky(x,t)=0,\ x=0$$

$$y(x,t)=0,\ EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}=0,\ x=L$$

RESPONSE TO EXTERNAL EXCITATIONS

$$y(x,t)=\sum_{r=1}^{\infty} Y_r(x)\eta_r(t)$$

Orthonormal modes

$$\int_0^L m(x)Y_r(x)Y_s(x)dx=\delta_{rs},\quad r,s=1,2,\ldots$$

$$\int_0^L Y_s(x)\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y_r(x)}{dx^2}\right]dx=\omega_r^2\delta_{rs}$$

$$\ddot{\eta}_r(t)+\omega_r^2\eta_r(t)=N_r(t),$$

$$N_r(t)=\int_0^L Y_r(x)f(x,t)dx$$

RESPONSE TO EXTERNAL EXCITATIONS: Harmonic Excitation

$$f(x,t) = F(x)\cos\Omega t$$

$$N_r(t) = \left[\int_0^L Y_r(x)F(x)dx\right]\cos\Omega t = F_r\cos\Omega t,$$

$$F_r = \int_0^L Y_r(x)F(x)dx,\ r = 1, 2, \ldots$$

Controls which mode is excited.

$$\eta_r(t) = \frac{F_r}{\omega_r^2 - \Omega^2}\cos\Omega t,$$

Controls the resonance.

$$y(x,t) = \left[\sum_{r=1}^{\infty}\frac{F_r}{\omega_r^2 - \Omega^2}Y_r(x)\right]\cos\Omega t$$

RESPONSE TO EXTERNAL EXCITATIONS: Arbitrary Excitation

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(t-\tau) \sin \omega_r \tau \, d\tau, \; r = 1, 2, \ldots$$

$$y(x,t) = \sum_{r=1}^{\infty} \frac{Y_r(x)}{\omega_r} \int_0^t N_r(t-\tau) \sin \omega_r \tau \, d\tau$$

The developments remain essentially the same for all other boundary conditions, and the same can be said about other systems.

Example

Derive the response of a uniform pinned-pinned beam to a concentrated force of amplitude F_0 acting at $x = L/2$ and having the form of a step function $f(x,t) = F_0\delta(x - L/2)\mathcal{u}(t)$

Orthonormal Modes

$$\omega_r = \beta_r^2\sqrt{\frac{EI}{m}} = (r\pi)^2\sqrt{\frac{EI}{mL^4}} \qquad Y_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L},\ r = 1, 2, \ldots$$

$$N_r(t) = \int_0^L Y_r(x) f(x,t)dx = \sqrt{\frac{2}{mL}}F_0\mathcal{u}(t)\int_0^L \sin\frac{r\pi x}{L}\delta(x - L/2)dx$$

$$= \sqrt{\frac{2}{mL}}F_0\mathcal{u}(t)\sin\frac{r\pi}{2} = (-1)^{(r-1)/2}\sqrt{\frac{2}{mL}}F_0\mathcal{u}(t),\ r = \text{odd}$$

Example

$$\eta_r(t)=\frac{1}{\omega_r}\int_0^t N_r(t-\tau)\sin\omega_r\tau\,d\tau=\frac{(-1)^{(r-1)/2}F_0}{\omega_r}\sqrt{\frac{2}{mL}}\int_0^t \boldsymbol{u}(t-\tau)\sin\omega_r\tau\,d\tau$$

$$=\frac{(-1)^{(r-1)/2}F_0}{\omega_r^2}\sqrt{\frac{2}{mL}}(1-\cos\omega_r t)$$

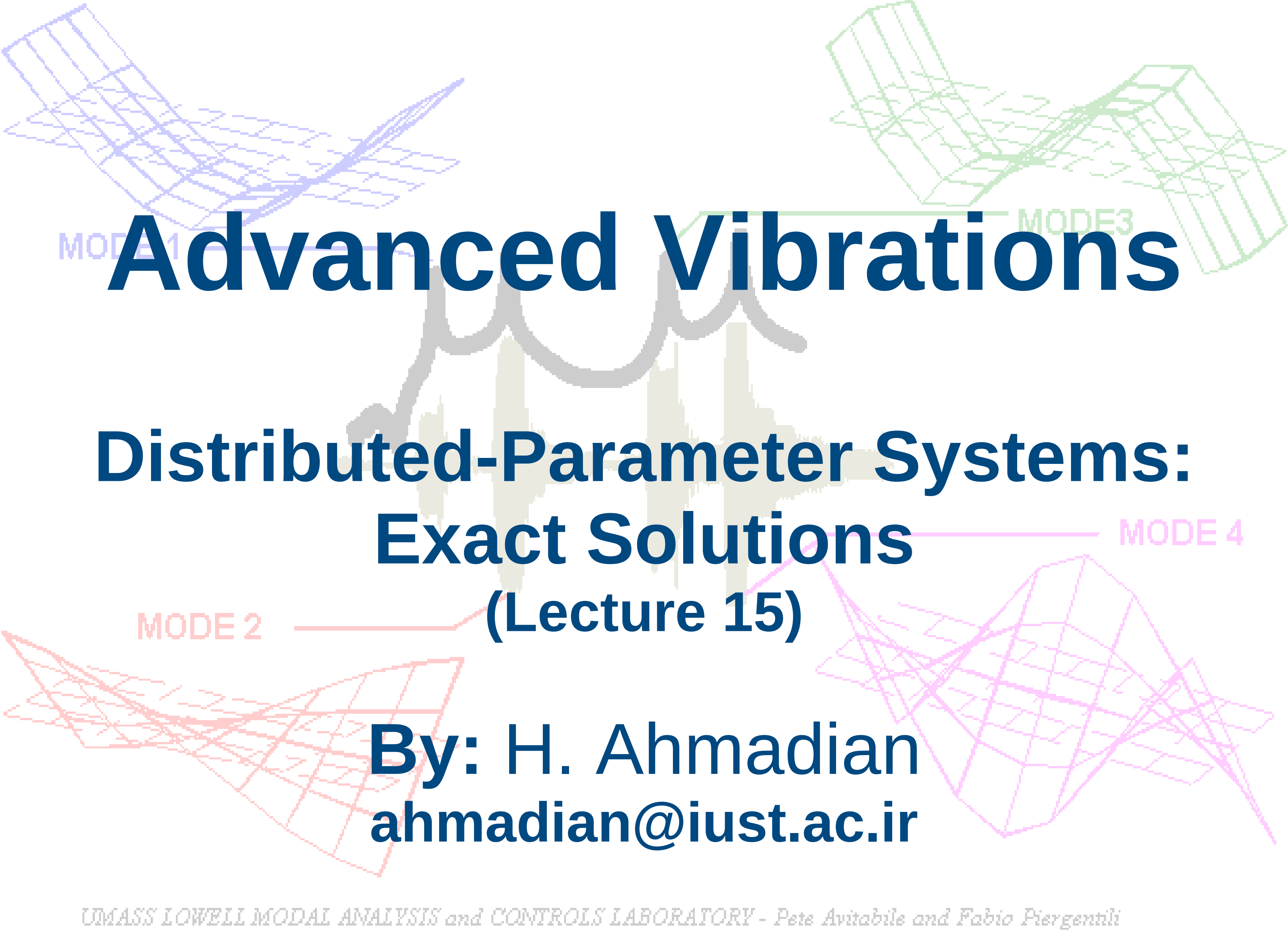
$$=\frac{(-1)^{(r-1)/2}F_0}{(r\pi)^4}\frac{mL^4}{EI}\sqrt{\frac{2}{mL}}\left[1-\cos(r\pi)^2\sqrt{\frac{EI}{mL^4}}t\right],\quad r=\text{odd}$$

$$y(x,t)=\sum_{r=1}^{\infty}Y_r(x)\eta_r(t)=\sum_{r=1,3,\ldots}^{\infty}\frac{(-1)^{(r-1)/2}F_0}{(r\pi)^4}\frac{mL^4}{EI}\frac{2}{mL}\sin\frac{r\pi x}{L}\left[1-\cos(r\pi)^2\sqrt{\frac{EI}{mL^4}}t\right]$$

$$=\frac{2F_0L^3}{\pi^4EI}\sum_{r=1,3,\ldots}^{\infty}\frac{(-1)^{(r-1)/2}}{r^4}\sin\frac{r\pi x}{L}\left[1-\cos(r\pi)^2\sqrt{\frac{EI}{mL^4}}t\right]$$

Distributed-Parameter Systems: Exact Solutions

- Relation between Discrete and Distributed Systems .
- Transverse Vibration of Strings
- Derivation of the String Vibration Problem by the Extended Hamilton Principle
- Bending Vibration of Beams
- Free Vibration: The Differential Eigenvalue Problem
- Orthogonality of Modes Expansion Theorem
- Systems with Lumped Masses at the Boundaries
- Eigenvalue Problem and Expansion Theorem for Problems with Lumped Masses at the Boundaries
- Rayleigh's Quotient . The Variational Approach to the Differential Eigenvalue Problem
- Response to Initial Excitations
- Response to External Excitations
- Systems with External Forces at Boundaries
- The Wave Equation
- Traveling Waves in Rods of Finite Length



Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 15)

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Stepped Beams

- Free Vibrations of Stepped Beams
  - Compatibility Requirements at the Interface
  - Characteristic Equations
- Elastically Restrained Stepped Beams
- Multi-Step Beam with Arbitrary Number of Cracks
- Multi-Step Beam Carrying a Tip Mass

FREE VIBRATION OF STEPPED BEAMS: EXACT SOLUTIONS

As presented by:

- S. K. JANG and C. W. BERT **1989** *Journal of Sound and Vibration* **130,** *342-346.* Free vibration of stepped beams: exact and numerical solutions.
- They sought lowest natural frequency of a stepped beam with two different cross-sections for various boundary conditions.

FREE VIBRATION OF STEPPED BEAMS: EXACT SOLUTIONS

The governing differential equation for the small amplitude, free, lateral vibration of a Bernoulli-Euler beam is:

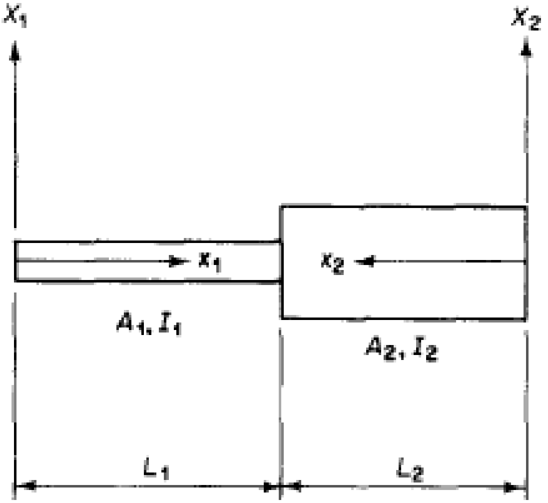
$$(\partial^2/\partial x^2)(EI(x)\,\partial^2 y/\partial x^2) = -\rho A(x)\,\partial^2 y/\partial t^2,$$

Assuming normal modes, one obtains the following expression for the mode shape:

$$(\mathrm{d}^2/\mathrm{d}x^2)(EI(x)\,\mathrm{d}^2 X/\mathrm{d}x^2) = \omega^2 \rho A(x) X.$$

FREE VIBRATION OF STEPPED BEAMS: EXACT SOLUTIONS

For the shown stepped beam, one can rewrite the governing equation as:



$$d^4X_i/dx_i^4 = K_i^4X_i$$

$$K_i^4 = (\rho A_i/EI_i)\omega^2 \text{ and } i = 1, 2.$$

$$X_1 = C_1 \sin K_1x_1 + C_2 \cos K_1x_1 + C_3 \sinh K_1x_1 + C_4 \cosh K_1x_1, \qquad 0 \leq x_1 \leq L_1,$$

$$X_2 = C_5 \sin K_2x_2 + C_6 \cos K_2x_2 + C_7 \sinh K_2x_2 + C_8 \cosh K_2x_2, \qquad 0 \leq x_2 \leq L_2.$$

Boundary Conditions:

(1) pinned-pinned,

$X_1 = I_1\,\mathrm{d}^2X_1/\mathrm{d}x_1^2 = 0$ at $x_1 = 0$, $X_2 = I_2\,\mathrm{d}^2X_2/\mathrm{d}x_2^2 = 0$ at $x_2 = 0$;

(2) clamped-clamped,

$X_1 = \mathrm{d}X_1/\mathrm{d}x_1 = 0$ at $x_1 = 0$, $X_2 = \mathrm{d}X_2/\mathrm{d}x_2 = 0$ at $x_2 = 0$;

(3) clamped-free,

$X_1 = \mathrm{d}X_1/\mathrm{d}x_1 = 0$ at $x_1 = 0$, $I_2\,\mathrm{d}^2X_2/\mathrm{d}x_2^2 = (\mathrm{d}/\mathrm{d}x_2)(I_2\,\mathrm{d}^2X_2/\mathrm{d}x_2^2) = 0$ at $x_2 = 0$;

(4) clamped-pinned,

$X_1 = \mathrm{d}X_1/\mathrm{d}x_1 = 0$ at $x_1 = 0$, $X_2 = I_2\,\mathrm{d}^2X_2/\mathrm{d}x_2^2 = 0$ at $x_2 = 0$.

Compatibility Requirements at the Interface

Stress concentration at the junction of the two parts of the beam is neglected.

At the junction, the continuity of deflection, slope, moment and shear force has to be preserved:

$$X_1(L_1) = X_2(L_2), \qquad \frac{\mathrm{d}X_1}{\mathrm{d}x_1}(L_1) = -\frac{\mathrm{d}X_2}{\mathrm{d}x_2}(L_2), \qquad I_1\frac{\mathrm{d}^2X_1}{\mathrm{d}x_1^2}(L_1) = I_2\frac{\mathrm{d}^2X_2}{\mathrm{d}x_2^2}(L_2),$$

$$\frac{\mathrm{d}}{\mathrm{d}x_1}\left(I_1\frac{\mathrm{d}^2X_1}{\mathrm{d}x_1^2}\right)(L_1) = -\frac{\mathrm{d}}{\mathrm{d}x_2}\left(I_2\frac{\mathrm{d}^2X_2}{\mathrm{d}x_2^2}\right)(L_2).$$

The clamped-clamped beam problem: Introducing the BCs

$$X_1 = \mathrm{d}X_1/\mathrm{d}x_1 = 0 \text{ at } x_1 = 0, \qquad X_2 = \mathrm{d}X_2/\mathrm{d}x_2 = 0 \quad \text{at } x_2 = 0;$$

$$X_1 = C_1 \sin K_1 x_1 + C_2 \cos K_1 x_1 + C_3 \sinh K_1 x_1 + C_4 \cosh K_1 x_1, \qquad 0 \leqslant x_1 \leqslant L_1,$$

$$X_2 = C_5 \sin K_2 x_2 + C_6 \cos K_2 x_2 + C_7 \sinh K_2 x_2 + C_8 \cosh K_2 x_2, \qquad 0 \leqslant x_2 \leqslant L_2.$$

Yields: C_3=-C_1, C_4=-C_2, C_7=-C_5 . C_8=-C_6

The clamped-clamped beam problem: Compatibility Requirements

Let:

$$\mathrm{S1} \equiv \sin K_1 L_1, \qquad \mathrm{S2} \equiv \sin K_2 L_2, \qquad \mathrm{C1} \equiv \cos K_1 L_1, \qquad \mathrm{C2} = \cos K_2 L_2,$$

$$\mathrm{SH1} \equiv \sinh K_1 L_1, \qquad \mathrm{SH2} \equiv \sinh K_2 L_2, \qquad \mathrm{CH1} \equiv \cosh K_1 L_1, \qquad \mathrm{CH2} \equiv \cosh K_2 L_2,$$

$$K \equiv K_2 / K_1, \qquad I \equiv I_2 / I_1 .$$

Then the compatibility requirements yield:

$$\begin{bmatrix} \mathrm{S1}-\mathrm{SH1} & \mathrm{C1}-\mathrm{CH1} & -\mathrm{S2}+\mathrm{SH2} & -\mathrm{C2}+\mathrm{CH2} \\ \mathrm{C1}-\mathrm{CH1} & -\mathrm{S1}-\mathrm{SH1} & K(\mathrm{C2}-\mathrm{CH2}) & -K(\mathrm{S2}+\mathrm{SH2}) \\ -\mathrm{S1}-\mathrm{SH1} & -\mathrm{C1}-\mathrm{CH1} & K^2 I(\mathrm{S2}+\mathrm{SH2}) & K^2 I(\mathrm{C2}+\mathrm{CH2}) \\ -\mathrm{C1}-\mathrm{CH1} & \mathrm{S1}-\mathrm{SH1} & -K^3 I(\mathrm{C2}+\mathrm{CH2}) & K^3 I(\mathrm{S2}-\mathrm{SH2}) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}.$$

Characteristic Equations for Other BCs

(1) pinned-pinned

$$\begin{vmatrix} S1 & SH1 & -S2 & -SH2 \\ C1 & CH1 & K\,C2 & K\,CH2 \\ -S1 & SH1 & K^2 I\,S2 & -K^2 I\,SH2 \\ -C1 & CH1 & -K^3 I\,C2 & K^3 I\,CH2 \end{vmatrix} = 0;$$

(2) clamped-free,

$$\begin{vmatrix} S1-SH1 & C1-CH1 & -S2-SH2 & -C2-CH2 \\ C1-CH1 & -S1-SH1 & K(C2+CH2) & K(-S2+SH2) \\ -S1-SH1 & -C1-CH1 & K^2 I(S2-SH2) & K^2 I(C2-CH2) \\ -C1-CH1 & S1-SH1 & -K^3 I(C2-CH2) & K^3 I(S2+CH2) \end{vmatrix} = 0;$$

Characteristic Equations for Other BCs

(3) clamped-pinned,

$$\begin{vmatrix} \mathrm{S1-SH1} & \mathrm{C1-CH1} & \mathrm{-S2} & \mathrm{-SH2} \\ \mathrm{C1-CH1} & \mathrm{-S1-SH1} & K\,\mathrm{C2} & K\,\mathrm{CH2} \\ \mathrm{-S1-SH1} & \mathrm{-C1-CH1} & K^{2}I\,\mathrm{S2} & -K^{2}I\,\mathrm{SH2} \\ \mathrm{-C1-CH1} & \mathrm{S1-SH1} & -K^{3}I\,\mathrm{C2} & K^{3}I\,\mathrm{CH2} \end{vmatrix} = 0;$$

(4) free-free

$$\begin{vmatrix} \mathrm{S1+SH1} & \mathrm{C1+CH1} & \mathrm{-S2-SH2} & \mathrm{-C2-CH2} \\ \mathrm{C1+CH1} & \mathrm{-S1+SH1} & K(\mathrm{C2+CH2}) & K(\mathrm{-S2+SH2}) \\ \mathrm{-S1+SH1} & \mathrm{-C1+CH1} & K^{2}I(\mathrm{S2-SH2}) & K^{2}I(\mathrm{C2-CH2}) \\ \mathrm{-C1+CH1} & \mathrm{S1+SH1} & -K^{3}I(\mathrm{C2-CH2}) & K^{3}I(\mathrm{S2+SH2}) \end{vmatrix} = 0;$$

Characteristic Equations for Other BCs

(5) sliding–sliding,

$$\begin{vmatrix} \mathrm{C1} & \mathrm{CH1} & -\mathrm{C2} & -\mathrm{C2} & -\mathrm{CH2} \\ -\mathrm{S1} & \mathrm{SH1} & -K\,\mathrm{S2} & -K\,\mathrm{S2} & K\,\mathrm{SH2} \\ -\mathrm{C1} & \mathrm{CH1} & K^2 I\,\mathrm{C2} & K^2 I\,\mathrm{C2} & -K^2 I\,\mathrm{CH2} \\ \mathrm{S1} & \mathrm{SH1} & K^3 I\,\mathrm{S2} & K^3 I\,\mathrm{S2} & K^3 I\,\mathrm{SH2} \end{vmatrix} = 0$$

(6) sliding–pinned,

$$\begin{vmatrix} \mathrm{C1} & \mathrm{CH1} & -\mathrm{S2} & -\mathrm{SH2} \\ -\mathrm{S1} & \mathrm{SH1} & K\,\mathrm{C2} & K\,\mathrm{CH2} \\ -\mathrm{C1} & \mathrm{CH1} & K^2 I\,\mathrm{S2} & -K^2 I\,\mathrm{SH2} \\ \mathrm{S1} & \mathrm{SH1} & -K^3 I\,\mathrm{C2} & K^3 I\,\mathrm{CH2} \end{vmatrix} = 0;$$

Characteristic Equations for Other BCs

(7) clamped-sliding,

$$\left|\begin{array}{cccc} S1-SH1 & C1-CH1 & -C2 & -CH2 \\ C1-CH1 & -S1-SH1 & -K\,S2 & K\,SH2 \\ -S1-SH1 & -C1-CH1 & K^{2}I\,C2 & -K^{2}I\,CH2 \\ -C1-CH1 & S1-SH1 & K^{3}I\,S2 & K^{2}I\,SH2 \end{array}\right| = 0;$$

(8) free-sliding,

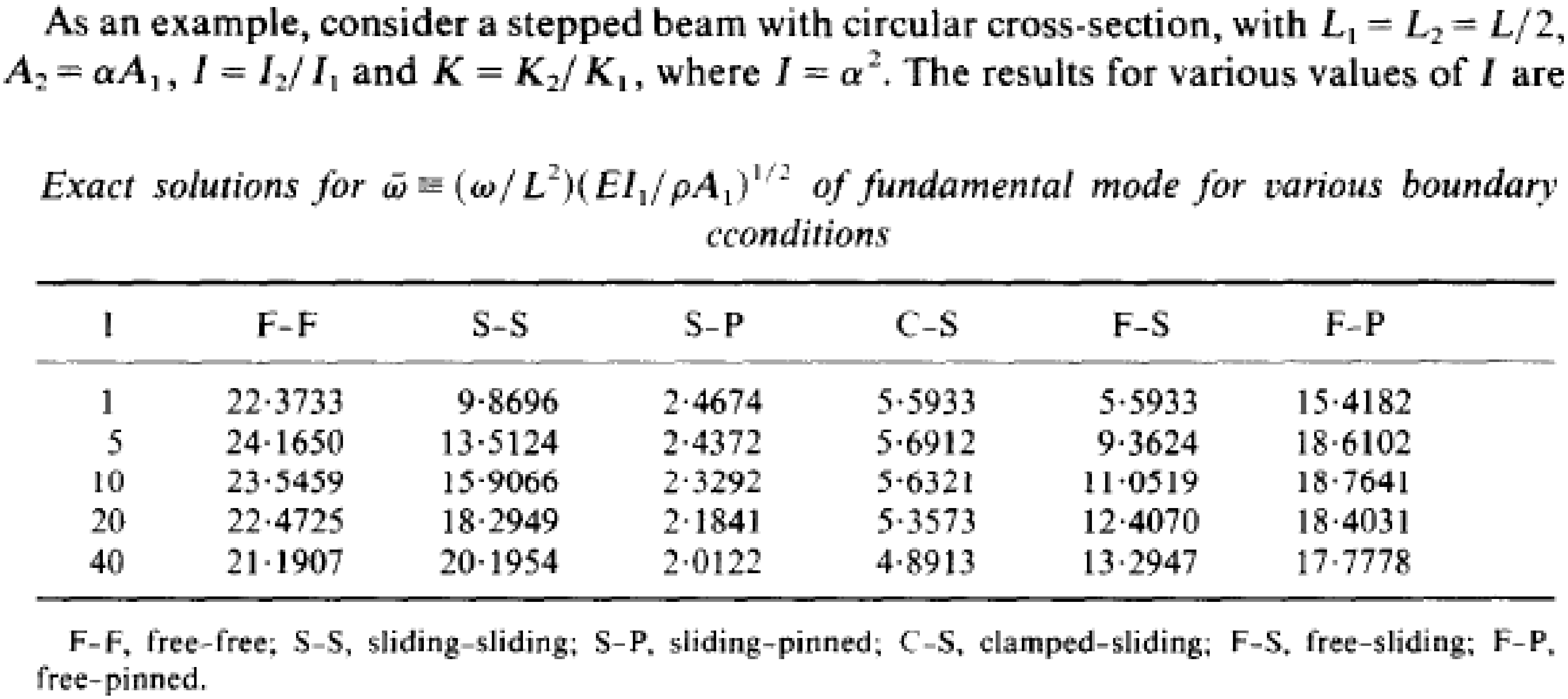
$$\left|\begin{array}{cccc} S1+SH1 & C1+CH1 & -C2 & -CH2 \\ C1+CH1 & -S1+SH1 & -K\,S2 & K\,SH2 \\ -S1+SH1 & -C1+CH1 & K^{2}I\,C2 & -K^{2}I\,CH2 \\ -C1+CH1 & S1+SH1 & K^{3}I\,S2 & K^{3}I\,SH2 \end{array}\right| = 0;$$

Characteristic Equations for Other BCs

(9) free–pinned,

$$\begin{vmatrix} \mathrm{S1}+\mathrm{SH1} & \mathrm{C1}+\mathrm{CH1} & -\mathrm{S2} & -\mathrm{SH2} \\ \mathrm{C1}+\mathrm{CH1} & -\mathrm{S1}+\mathrm{SH1} & K\,\mathrm{C2} & K\,\mathrm{CH2} \\ -\mathrm{S1}+\mathrm{SH1} & -\mathrm{C1}+\mathrm{CH1} & K^2 I\,\mathrm{S2} & -K^2 I\,\mathrm{SH2} \\ -\mathrm{C1}+\mathrm{CH1} & \mathrm{S1}+\mathrm{SH1} & -K^3 I\,\mathrm{C2} & K^3 I\,\mathrm{CH2} \end{vmatrix} = 0.$$

Exact Solutions:



As an example, consider a stepped beam with circular cross-section, with $L_1 = L_2 = L/2$, $A_2 = \alpha A_1$, $I = I_2/I_1$ and $K = K_2/K_1$, where $I = \alpha^2$. The results for various values of I are

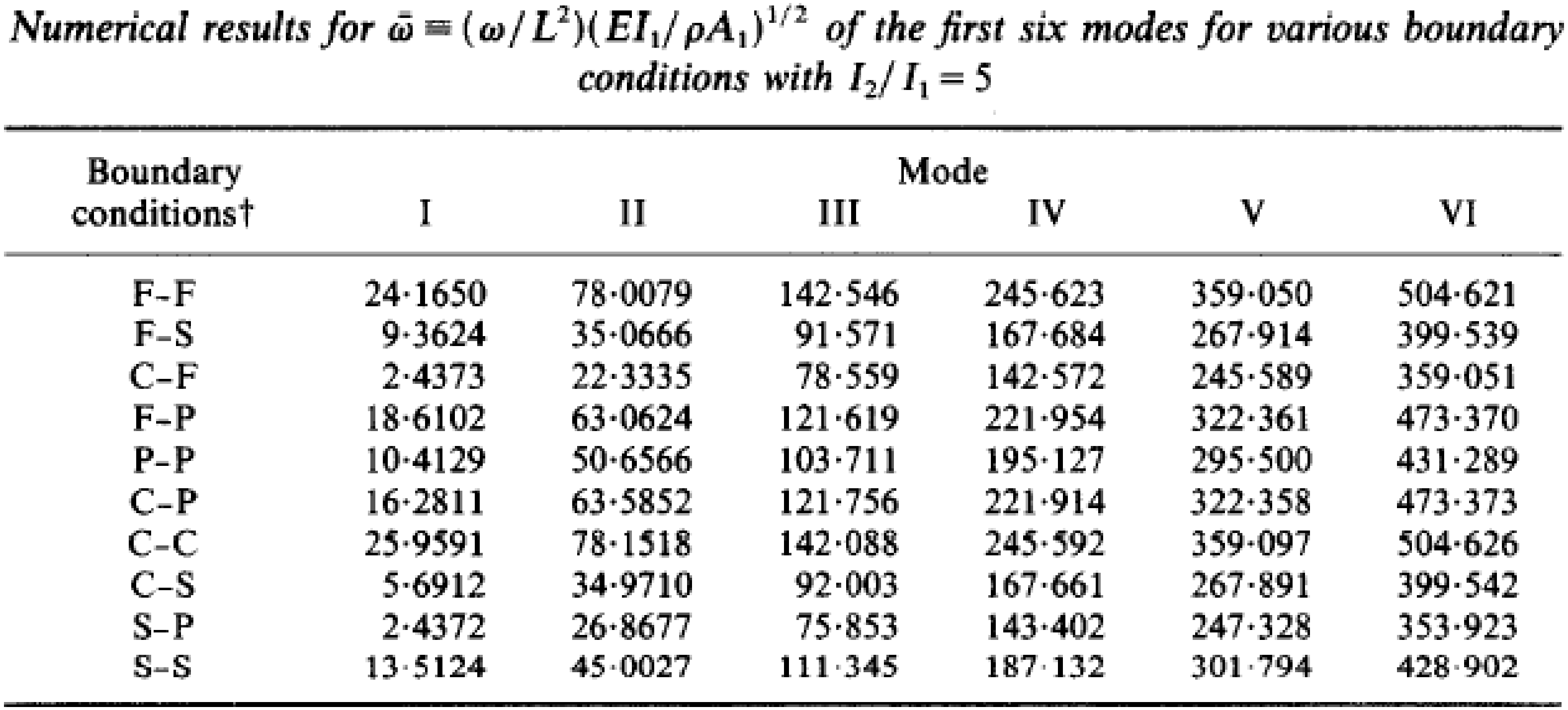
Exact solutions for $\bar{\omega} = (\omega/L^2)(EI_1/\rho A_1)^{1/2}$ of fundamental mode for various boundary cconditions

I	F–F	S–S	S–P	C–S	F–S	F–P
1	22·3733	9·8696	2·4674	5·5933	5·5933	15·4182
5	24·1650	13·5124	2·4372	5·6912	9·3624	18·6102
10	23·5459	15·9066	2·3292	5·6321	11·0519	18·7641
20	22·4725	18·2949	2·1841	5·3573	12·4070	18·4031
40	21·1907	20·1954	2·0122	4·8913	13·2947	17·7778

F–F, free–free; S–S, sliding–sliding; S–P, sliding–pinned; C–S, clamped–sliding; F–S, free–sliding; F–P, free–pinned.

HIGHER MODE FREQUENCIES AND EFFECTS OF STEPS ON FREQUENCY

By extending the computations, higher mode frequencies were found (Journal of Sound and Vibration ,1989, 132(1), 164-168):



Numerical results for $\bar{\omega} = (\omega/L^2)(EI_1/\rho A_1)^{1/2}$ *of the first six modes for various boundary conditions with* $I_2/I_1 = 5$

Boundary conditions†	Mode					
	I	II	III	IV	V	VI
F-F	24·1650	78·0079	142·546	245·623	359·050	504·621
F-S	9·3624	35·0666	91·571	167·684	267·914	399·539
C-F	2·4373	22·3335	78·559	142·572	245·589	359·051
F-P	18·6102	63·0624	121·619	221·954	322·361	473·370
P-P	10·4129	50·6566	103·711	195·127	295·500	431·289
C-P	16·2811	63·5852	121·756	221·914	322·358	473·373
C-C	25·9591	78·1518	142·088	245·592	359·097	504·626
C-S	5·6912	34·9710	92·003	167·661	267·891	399·542
S-P	2·4372	26·8677	75·853	143·402	247·328	353·923
S-S	13·5124	45·0027	111·345	187·132	301·794	428·902

Elastically Restrained Stepped Beams

FREE VIBRATION OF STEPPED BEAMS ELASTICALLY RESTRAINED AGAINST TRANSLATION AND ROTATION AT ONE END

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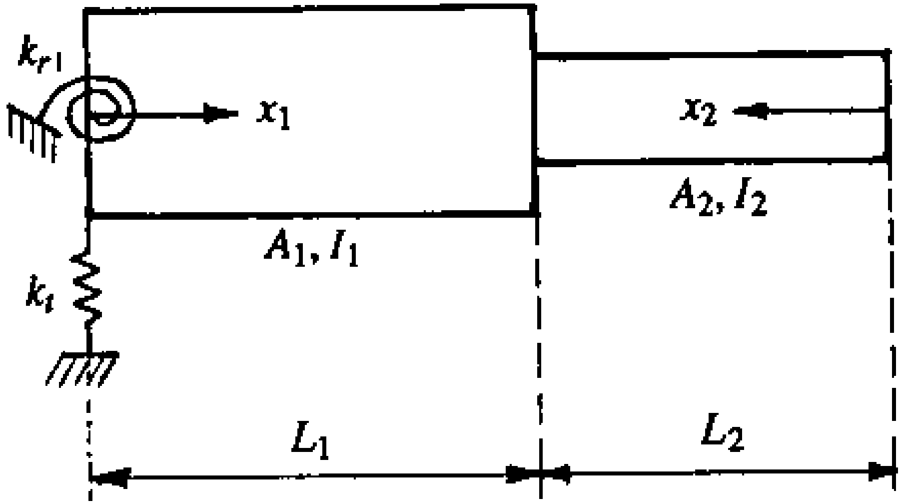
Journal of Sound and Vibration (1993) **163**(1), 188–191



Boundary Conditions and Compatibility Requirements at the Interface :

$x_1 = 0,$

$$k_r \, \mathrm{d}Y_1/\mathrm{d}x_1 = EI_1 \, \mathrm{d}^2 Y_1/\mathrm{d}x_1^2, \qquad EI_1 \, \mathrm{d}^3 Y_1/\mathrm{d}x_1^3 = -k_t Y_1 \, ;$$

at $x_2 = 0,$

$$EI_2 \, \mathrm{d}^2 Y_2/\mathrm{d}x_2^2 = 0, \qquad EI_2 \, \mathrm{d}^3 Y_2/\mathrm{d}x_2^3 = 0.$$

$$Y_1(L_1) = Y_2(L_2), \qquad \frac{\mathrm{d}Y_1}{\mathrm{d}x_1}(L_1) = -\frac{\mathrm{d}Y_2}{\mathrm{d}x_2}(L_2),$$

$$I_1 \frac{\mathrm{d}^2 Y_1}{\mathrm{d}x_1^2}(L_1) = I_2 \frac{\mathrm{d}^2 Y_2}{\mathrm{d}x_2^2}(L_2), \qquad \frac{\mathrm{d}}{\mathrm{d}x_1}\left(I_1 \frac{\mathrm{d}^2 Y_1}{\mathrm{d}x_1^2}\right)(L_1) = -\frac{\mathrm{d}}{\mathrm{d}x_2}\left(I_2 \frac{\mathrm{d}^2 Y_2}{\mathrm{d}x_2^2}\right)(L_2).$$

The Characteristic Equation

$$\begin{vmatrix} 1 & R(K_1L_1) & 1 & -R(K_1L_1) & 0 & 0 \\ -T(K_1L_1)^3 & 1 & T(K_1L_1)^3 & 1 & 0 & 0 \\ S1 & C1 & SH1 & CH1 & -(S2+SH2) & -(C2+CH2) \\ C1 & -S1 & CH1 & SH1 & K(C2+CH2) & K(-S2+SH2) \\ -S1 & -C1 & SH1 & CH1 & -K^2I(-S2+SH2) & -K^2I(-C2+CH2) \\ -C1 & S1 & CH1 & SH1 & K^3I(-C2+CH2) & K^3I(S2+SH2) \end{vmatrix} = 0.$$

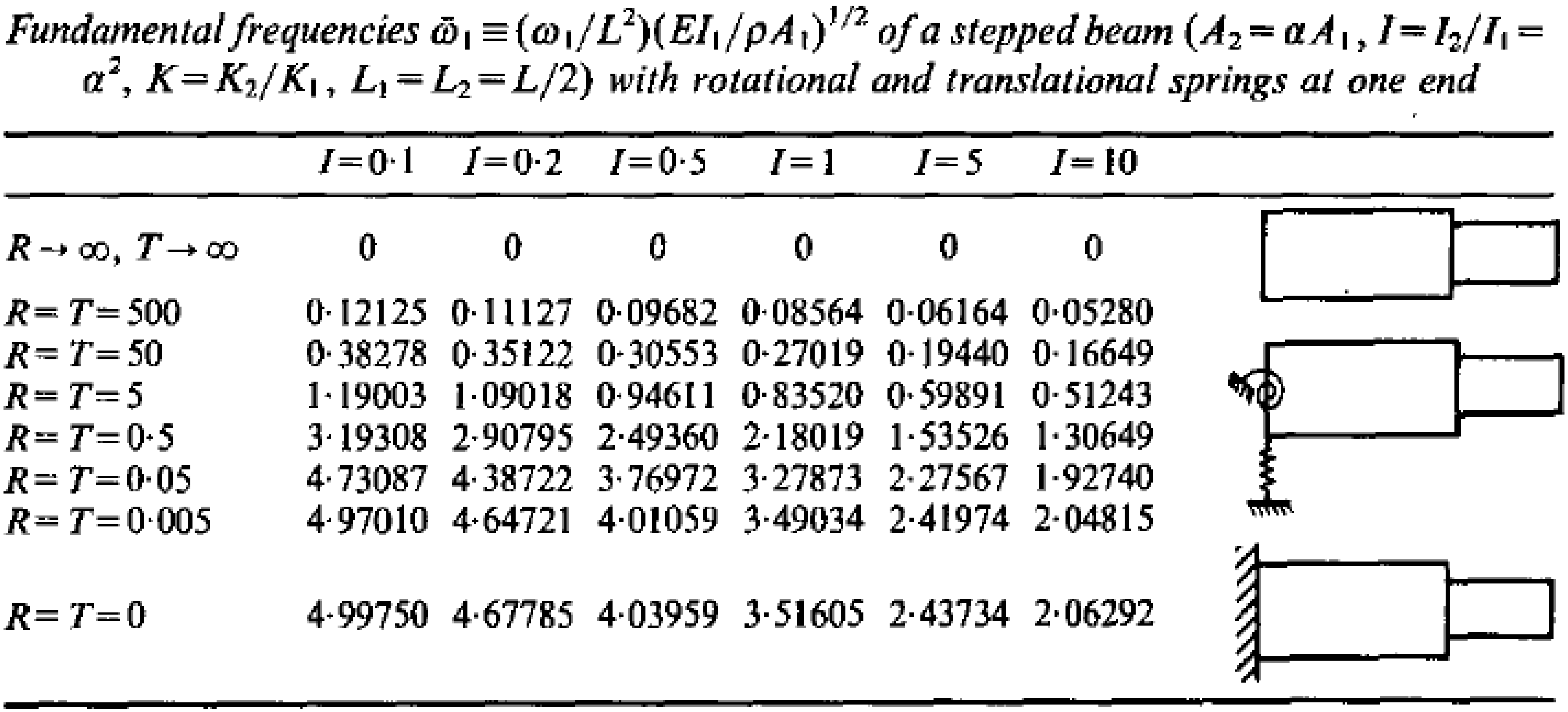
Here

$$S1 = \sin K_1L_1, \qquad S2 = \sin K_2L_2, \qquad Cl = \cos K_1L_1, \qquad C2 = \cos K_2L_2,$$

$$SH1 = \sinh K_1L_1, \qquad SH2 = \sinh K_2L_2, \qquad CH1 = \cosh K_1L_1, \qquad CH2 = \cosh K_2L_2;$$

$$K = K_2/K_1, \qquad I = I_2/I_1, \qquad R = EI_1/k_rL_1, \qquad T = EI_1/k_rL_1^3.$$

Exact Solutions:



Fundamental frequencies $\bar{\omega}_1 \equiv (\omega_1/L^2)(EI_1/\rho A_1)^{1/2}$ *of a stepped beam* ($A_2 = \alpha A_1$, $I = I_2/I_1 = \alpha^2$, $K = K_2/K_1$, $L_1 = L_2 = L/2$) *with rotational and translational springs at one end*

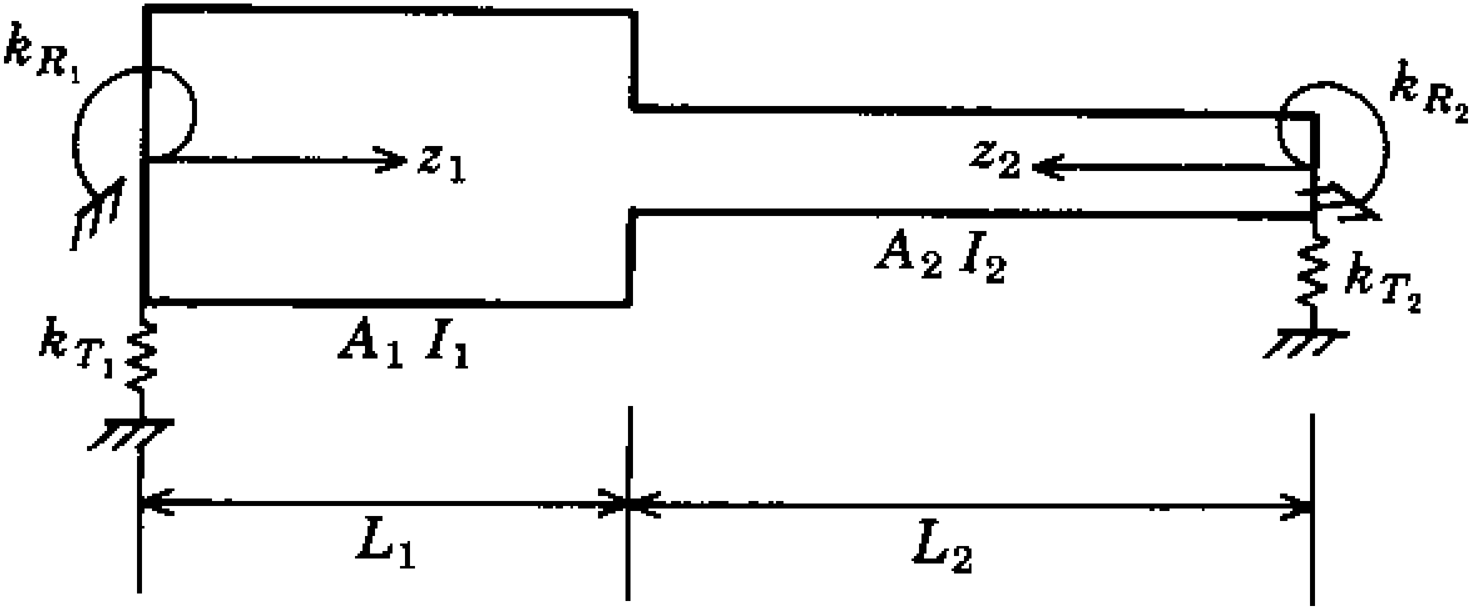
	$I=0·1$	$I=0·2$	$I=0·5$	$I=1$	$I=5$	$I=10$
$R \to \infty$, $T \to \infty$	0	0	0	0	0	0
$R=T=500$	0·12125	0·11127	0·09682	0·08564	0·06164	0·05280
$R=T=50$	0·38278	0·35122	0·30553	0·27019	0·19440	0·16649
$R=T=5$	1·19003	1·09018	0·94611	0·83520	0·59891	0·51243
$R=T=0·5$	3·19308	2·90795	2·49360	2·18019	1·53526	1·30649
$R=T=0·05$	4·73087	4·38722	3·76972	3·27873	2·27567	1·92740
$R=T=0·005$	4·97010	4·64721	4·01059	3·49034	2·41974	2·04815
$R=T=0$	4·99750	4·67785	4·03959	3·51605	2·43734	2·06292

Extension of the Research Work:

FREE VIBRATIONS OF STEPPED BEAMS WITH ELASTIC ENDS

M. A. DE ROSA

Journal of Sound and Vibration (1994) **173**(4), 563–567

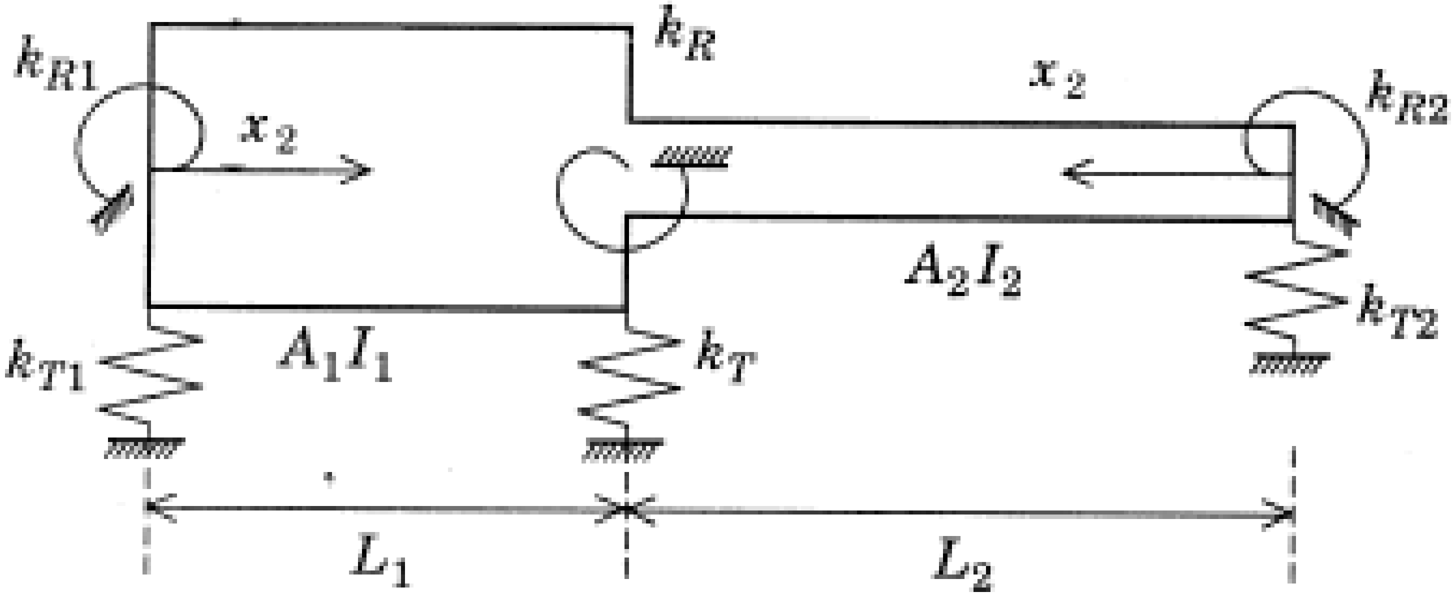


Extension of the Research Work:

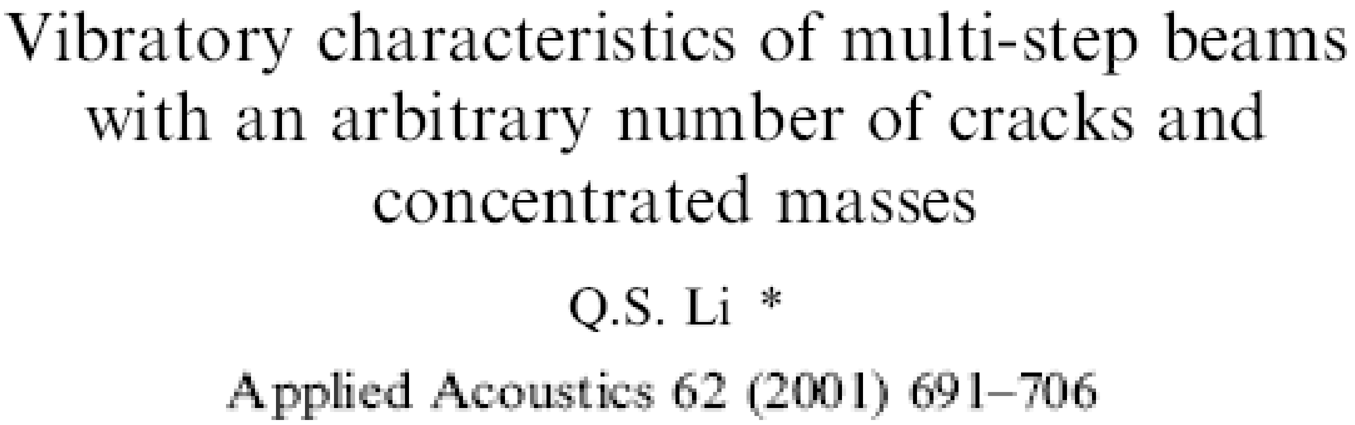
FREE VIBRATIONS OF STEPPED BEAMS WITH INTERMEDIATE ELASTIC SUPPORTS

M. A. DE ROSA

Journal of Sound and Vibration (1995) **181**(5), 905–910



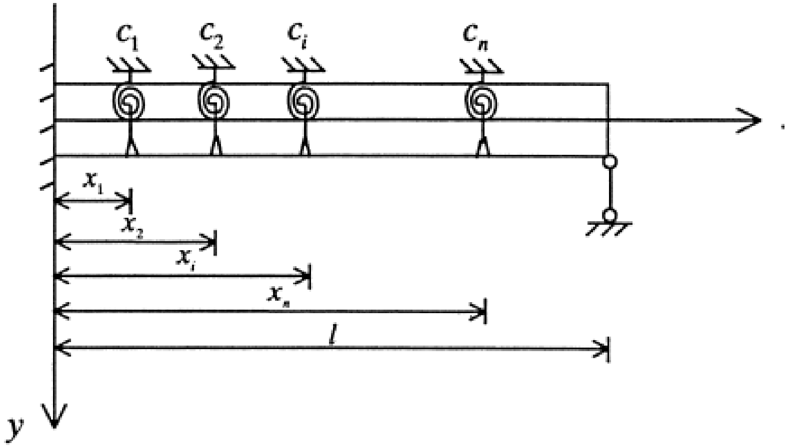
Extension of the Research Work:

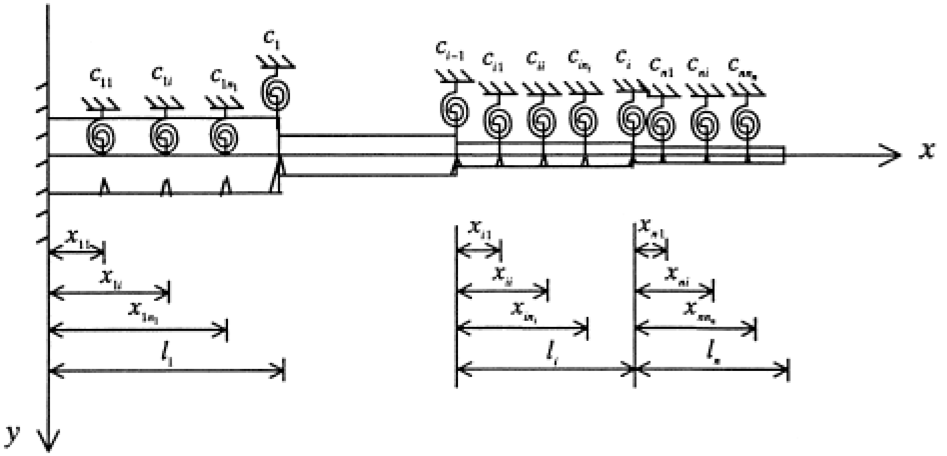


Vibratory characteristics of multi-step beams with an arbitrary number of cracks and concentrated masses

Q.S. Li *

Applied Acoustics 62 (2001) 691–706



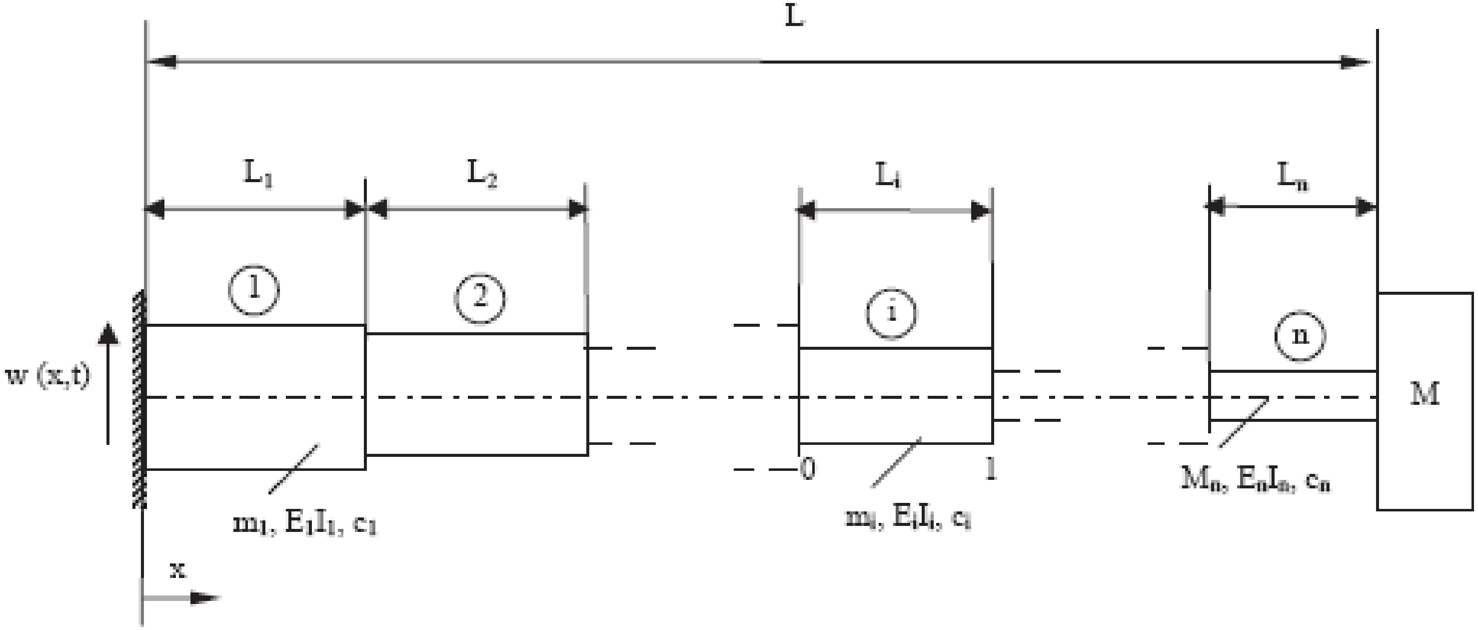


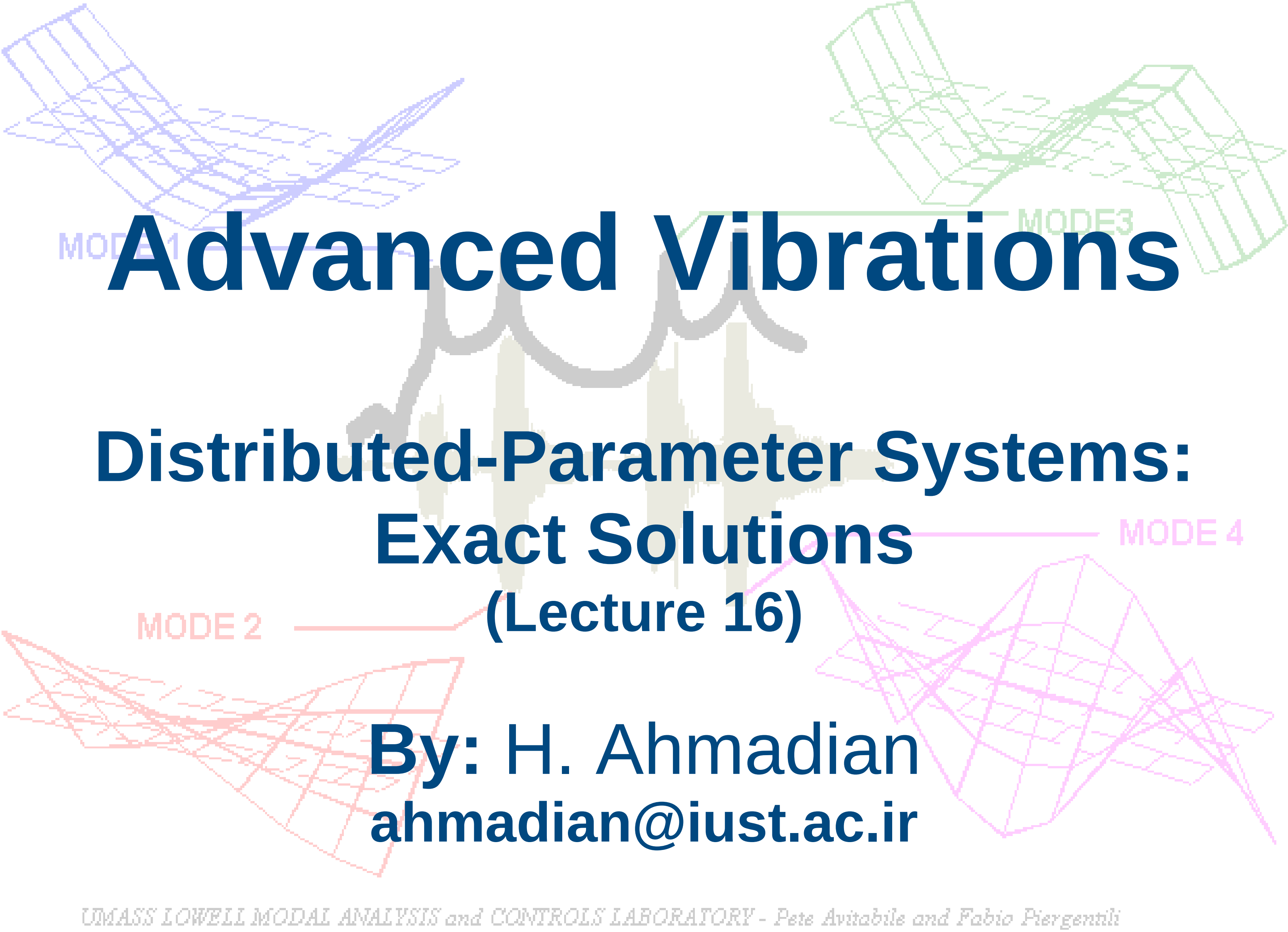
Extension of the Research Work:

On the eigencharacteristics of multi-step beams carrying a tip mass subjected to non-homogeneous external viscous damping

M. Gürgöze*, H. Erol

Journal of Sound and Vibration 272 (2004) 1113–1124





Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 16)

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INTRODUCTION

The problem of lateral vibrations of beams under axial loading is of considerable practical interest,

- Tall buildings
- Aerospace structures
- Rotating machinery shafts

Because of its important practical applications, the problem of uniform single-span beams under a constant axial load has been the subject of considerable study.

BEAM FLEXURE: INCLUDING AXIAL-FORCE EFFECTS

Axial forces acting in a flexural element may have a very significant influence on the vibration behavior of the member,

- resulting generally in modifications of frequencies and mode shapes.

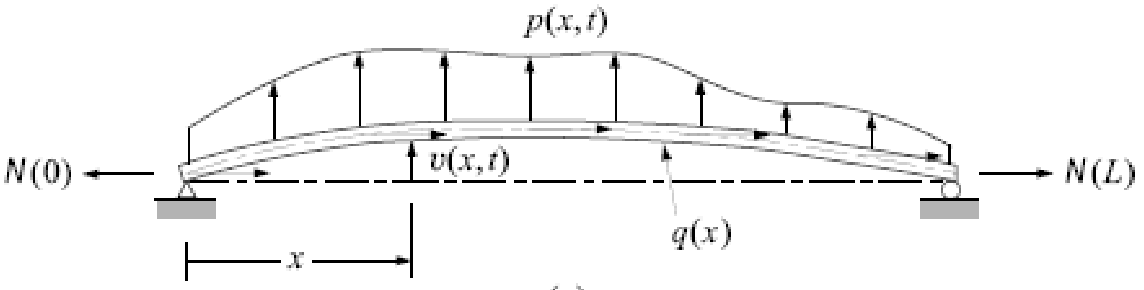
The equation of motion, including the effect of a time-invariant uniform axial force throughout its length, is:

$$EI\frac{\partial^4 v(x,t)}{\partial x^4} + N\frac{\partial^2 v(x,t)}{\partial x^2} + \overline{m}\frac{\partial^2 v(x,t)}{\partial t^2} = 0$$

BEAM FLEXURE: INCLUDING AXIAL-FORCE EFFECTS

$$\frac{\partial V(x,t)}{\partial x} = p(x,t) - m(x)\frac{\partial^2 v(x,t)}{\partial t^2}$$

$$V(x,t) = -N(x)\frac{\partial v(x,t)}{\partial x} + \frac{\partial M(x,t)}{\partial x}$$

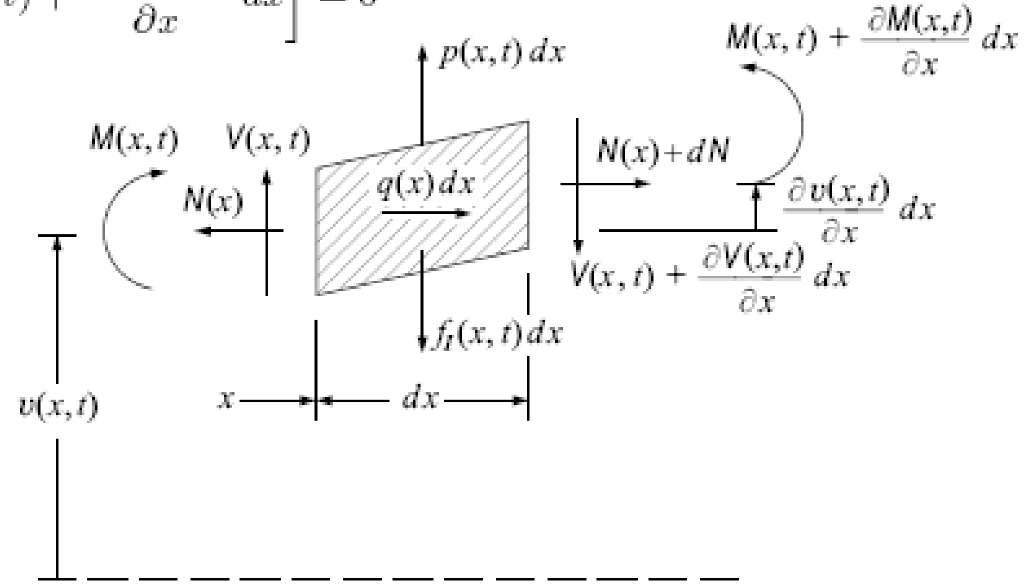


(a)

$$M(x,t) + V(x,t)\,dx + N(x)\frac{\partial v(x,t)}{\partial x}\,dx - \left[M(x,t) + \frac{\partial M(x,t)}{\partial x}\,dx\right] = 0$$

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 v(x,t)}{\partial x^2}\right] - \frac{\partial}{\partial x}\left[N(x)\frac{\partial v(x,t)}{\partial x}\right].$$

$$+m(x)\frac{\partial^2 v(x,t)}{\partial t^2} = p(x,t)$$



BEAM FLEXURE: INCLUDING AXIAL-FORCE EFFECTS

Separating variables:

$$\frac{\phi^{iv}(x)}{\phi(x)} + \frac{N}{EI}\,\frac{\phi''(x)}{\phi(x)} = -\frac{\overline{m}}{EI}\,\frac{\ddot{Y}(t)}{Y(t)} = a^4$$

$$\ddot{Y}(t) + \omega^2 Y(t) = 0$$

$$\phi^{iv}(x) + g^2\,\phi''(x) - a^4\,\phi(x) = 0 \qquad g^2 \equiv \frac{N}{EI}$$

$$(s^4 + g^2 s^2 - a^4)\,G\exp(sx) = 0$$

BEAM FLEXURE: INCLUDING AXIAL-FORCE EFFECTS

$$(s^4 + g^2 s^2 - a^4)\, G \exp(sx) = 0$$

$$s_{1,2} = \pm i\delta \qquad s_{3,4} = \pm \epsilon$$

$$\delta \equiv \sqrt{\left(a^4 + \frac{g^4}{4}\right)^{1/2} + \frac{g^2}{2}} \qquad \epsilon \equiv \sqrt{\left(a^4 + \frac{g^4}{4}\right)^{1/2} - \frac{g^2}{2}}$$

$$\phi(x) = D_1\ \cos\delta x + D_2\ \sin\delta x + D_3\ \cosh\epsilon x + D_4\ \sinh\epsilon x$$

Example: A simply supported uniform beam

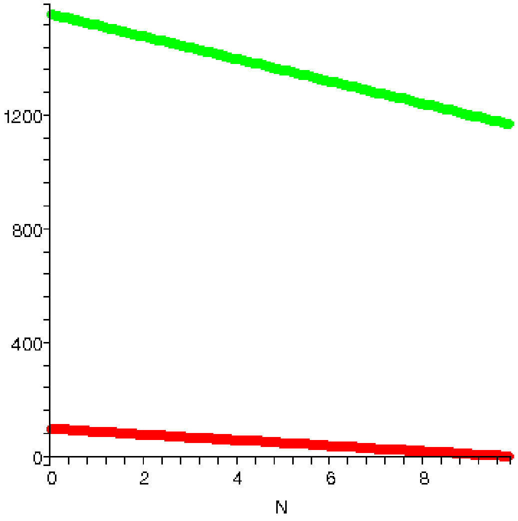
$$\phi(x) = D_1 \cos\delta x + D_2 \sin\delta x + D_3 \cosh\epsilon x + D_4 \sinh\epsilon x$$

$D_1=0$, $D_3=0$, $D_4=0$. $\phi(x) = D_2 \sin\delta x$

$$\delta = \frac{n\pi}{L}$$

I

$$a^4 = \frac{n^2\pi^2\left(-g^2L^2+n^2\pi^2\right)}{L^4}$$



BEAM FLEXURE: INCLUDING AXIAL-FORCE EFFECTS

Retaining the constant axial force *N*, the governing equation can be used to find the static buckling loads and corresponding shapes:

$$\omega = 0 \xrightarrow{\qquad\qquad} a = 0,\ \delta = g, \quad \epsilon = 0,$$

$$\phi(x) = D_1 \cos gx + D_2 \sin gx + D_3\, x + D_4$$

GALEF Formula

- E. GALEF 1968 ***Journal of the Acoustical Society of America 44, (8), 643.*** Bending frequencies of compressed beams:

$$\frac{\omega^{2}}{\omega 0^{2}} = 1 - \frac{N}{Ncr}$$

GALEF Formula

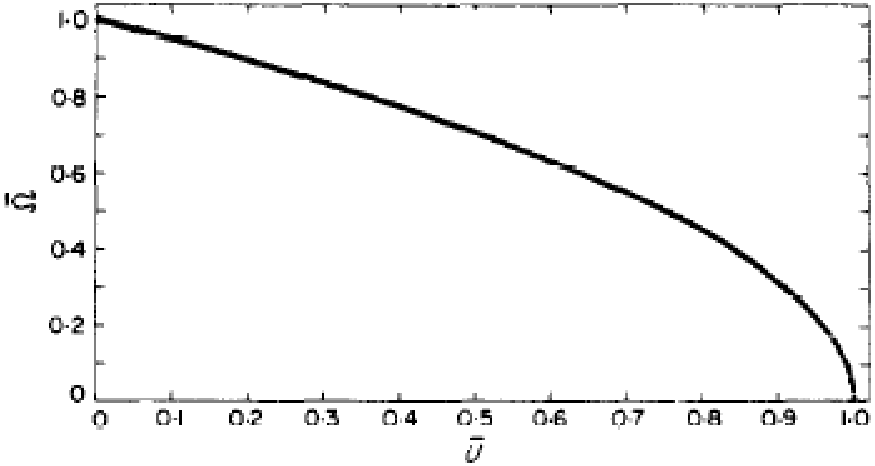
A. **BOKAIAN,** “NATURAL FREQUENCIES OF BEAMS UNDERCOMPRESSIVE AXIAL LOADS”, ***Journal of Sound and Vibration (1988) 126(1), 49-65***

- Studied the influence of a constant compressive load on natural frequencies and mode shapes of a uniform beam with a variety of end conditions.
- Galef’s formula, previously assumed to be valid for beams with all types of end conditions, is observed to be valid only for a few.

GALEF Formula

BOKAIAN showed:

- The variation of the normalized natural frequency $\bar{\Omega}$ with the normalized axial force $\bar{U}$ for pinned-pinned, pinned-sliding and sliding-sliding beams is observed to be $\bar{\Omega} = \sqrt{1 - \bar{U}}$.

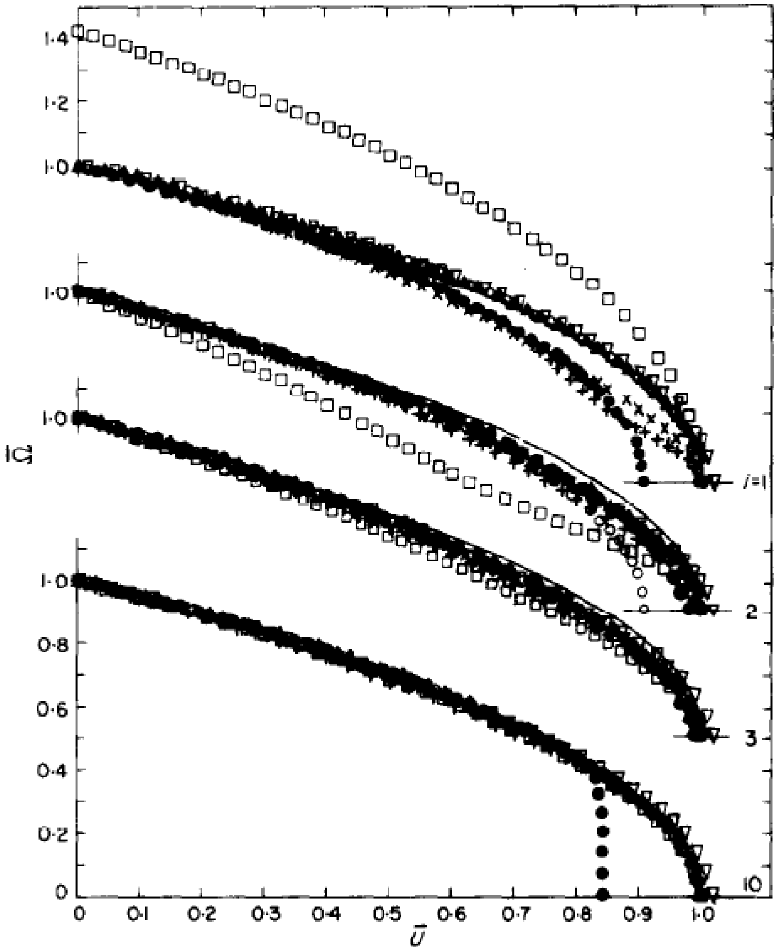


Variation of $\bar{\Omega}$ with $\bar{U}$ for a pinned-pinned or a pinned-sliding or a sliding-sliding beam.

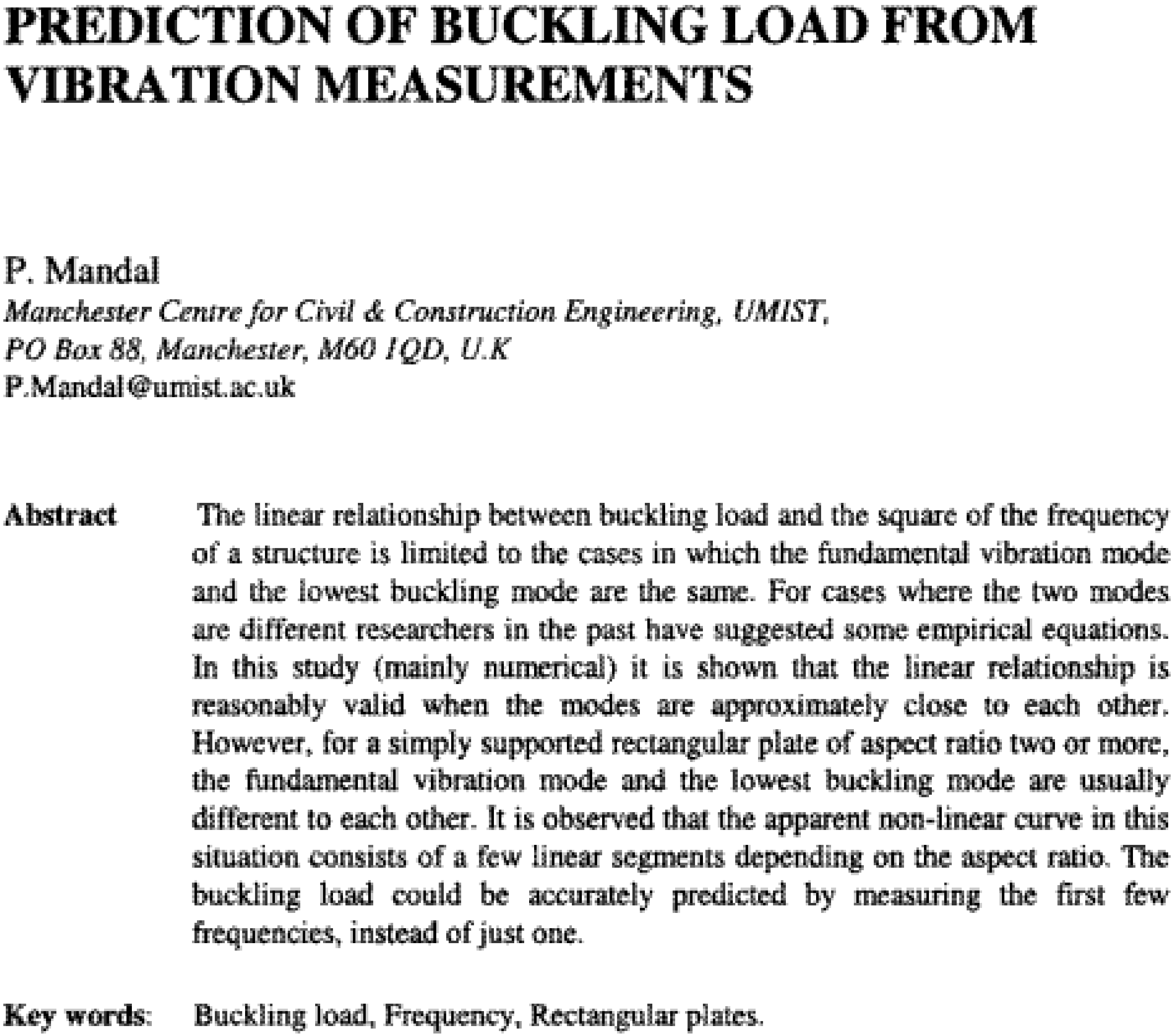
GALEF Formula

BOKAIAN showed:

- Galef’s formula, previously assumed to be valid for beams with all types of end conditions, is observed to be valid only for a few.
- The effect of end constraints on natural frequency of a beam is significant only in the first few modes.



Variation of $\bar{\Omega}$ with $\bar{U}$ for the first, second, third and the tenth mode. ▽, Sliding-free; ×, pinned-free; ●, clamped-pinned; ○, clamped-clamped; □, clamped-free; ▲, clamped-sliding; +, free-free; ——, pinned-pinned or sliding-pinned or sliding-sliding.



PREDICTION OF BUCKLING LOAD FROM VIBRATION MEASUREMENTS

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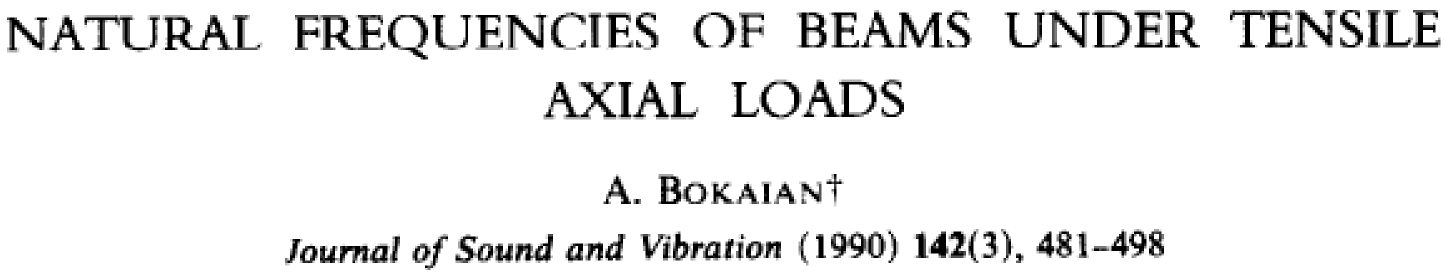
Abstract The linear relationship between buckling load and the square of the frequency of a structure is limited to the cases in which the fundamental vibration mode and the lowest buckling mode are the same. For cases where the two modes are different researchers in the past have suggested some empirical equations. In this study (mainly numerical) it is shown that the linear relationship is reasonably valid when the modes are approximately close to each other. However, for a simply supported rectangular plate of aspect ratio two or more, the fundamental vibration mode and the lowest buckling mode are usually different to each other. It is observed that the apparent non-linear curve in this situation consists of a few linear segments depending on the aspect ratio. The buckling load could be accurately predicted by measuring the first few frequencies, instead of just one.

Key words: Buckling load, Frequency, Rectangular plates.



NATURAL FREQUENCIES OF BEAMS UNDER TENSILE AXIAL LOADS

A. BOKAIAN†

Journal of Sound and Vibration (1990) **142**(3), 481–498

- For pinned-pinned, pinned-sliding and sliding-sliding beams, this variation may exactly be expressed as $\bar{\Omega} = \sqrt{1+\bar{U}}$
- This formula may be used for beams with other types of end constraints when the beam vibrates in a third mode or higher.
- For beam with other types of boundary conditions, this approximation may be expressed as $\bar{\Omega} = \sqrt{1+\gamma\bar{U}}\ (\gamma < 1)$ where the coefficient y depends only on the type of the end constraints.

NATURAL FREQUENCIES OF BEAMS UNDER TENSILE AXIAL LOADS

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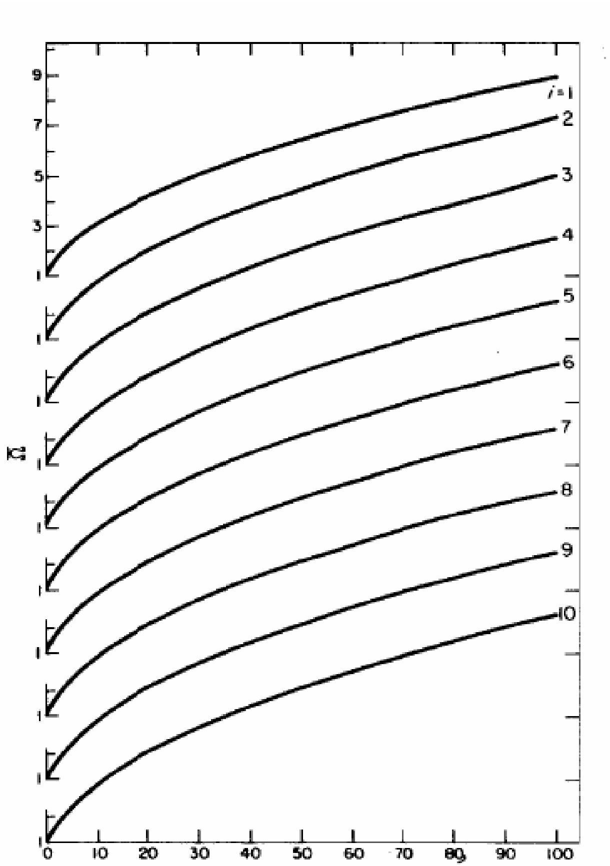
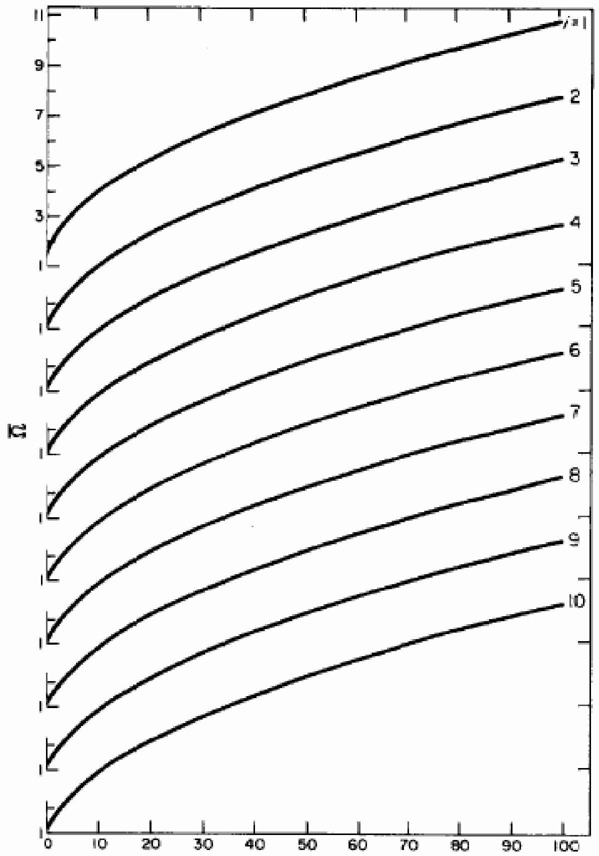
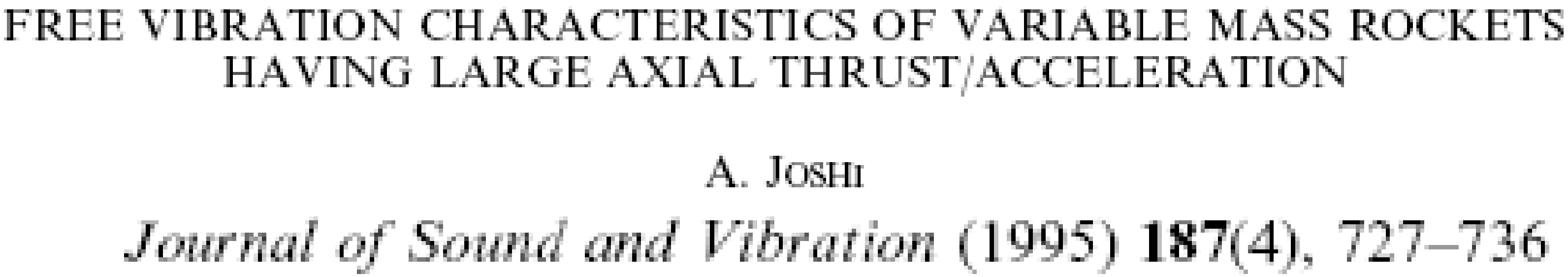
Journal of Sound and Vibration (1990) **142**(3), 481–498



Variation of $\bar{\Omega}$ with $\bar{U}$ for a sliding-free beam.



Variation of $\bar{\Omega}$ with $\bar{U}$ for a clamped-free beam.



FREE VIBRATION CHARACTERISTICS OF VARIABLE MASS ROCKETS HAVING LARGE AXIAL THRUST/ACCELERATION

A. JOSHI

Journal of Sound and Vibration (1995) **187**(4), 727–736

The study is an investigation of the combined effects of

- compressive inertia forces due to a conservative model of steady thrust and
- uniform mass depletion on the transverse vibration characteristics of a single stage variable mass rocket.
- the effect of the aerodynamic drag in comparison to the thrust is considered to be negligible and
- the rocket is structurally modeled as a non-uniform slender beam representative of practical rocket configurations.

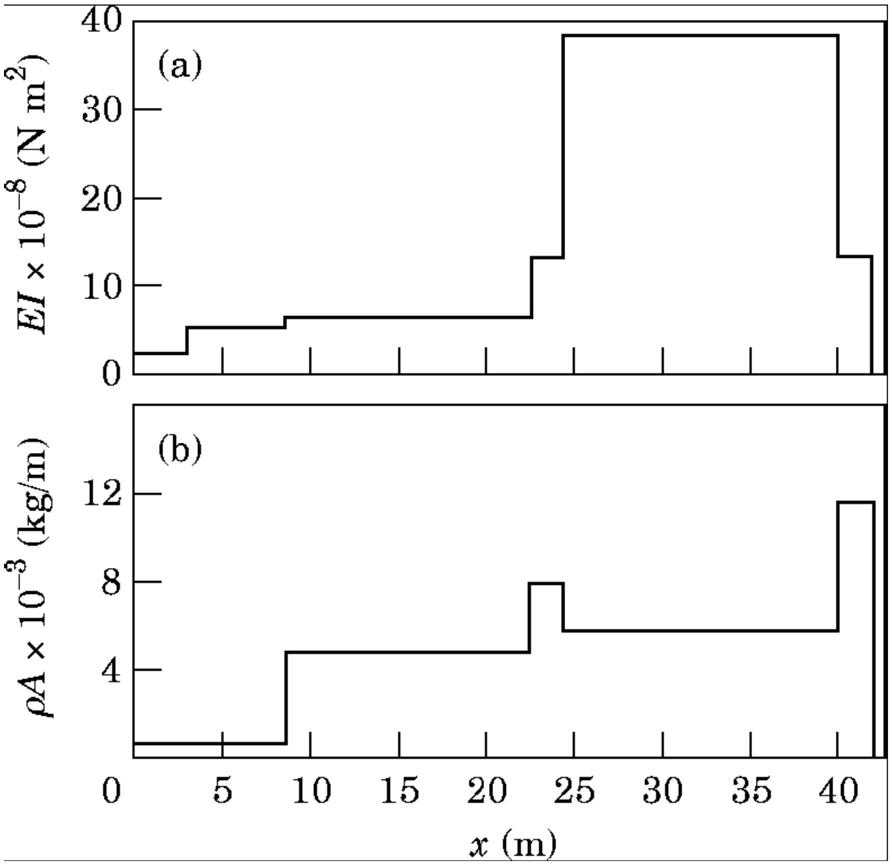
FREE VIBRATION CHARACTERISTICS OF VARIABLE MASS ROCKETS HAVING LARGE AXIAL THRUST/ACCELERATION

A. JOSHI

Journal of Sound and Vibration (1995) **187**(4), 727–736

In the study the typical single stage rocket structure is divided into a number of segments.

- Within which the bending rigidity, axial compressive force and the mass distributions can be approximated as constants.



FREE VIBRATION CHARACTERISTICS OF VARIABLE MASS ROCKETS
HAVING LARGE AXIAL THRUST/ACCELERATION

A. JOSHI

Journal of Sound and Vibration (1995) **187**(4), 727–736

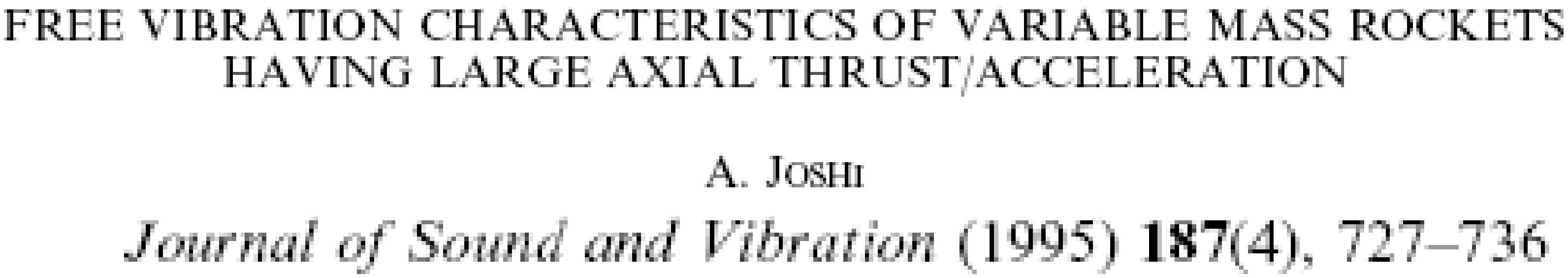
The non-dimensional equation of motion for the *i*th constant beam segment :

$$(\partial^4 w_i/\partial \bar{x}_i^4) - a_i(\partial^2 w_i/\partial \bar{x}_i^2) + \lambda_i^4 w_i = 0.$$

$a_i \quad \{= P(x_i)L_0^2(EI)_i\}$ $\qquad \lambda_i\{=(\rho A)_i\omega^2 L_0^4/(EI)_i\}^{(1/4)}$

$$w_i = A_i \cosh \lambda_1 \bar{x}_i + B_i \sinh \lambda_1 \bar{x}_i + C_i \cos \lambda_2 \bar{x}_i + D_i \sin \lambda_2 \bar{x}_i,$$

$$\lambda_1^2 = \{(a_i^2 + 4\lambda_i^4)^{1/2} - a_i\}/2, \qquad \lambda_2^2 = \{(a_i^2 + 4\lambda_i^4)^{1/2} + a_i\}/2.$$



FREE VIBRATION CHARACTERISTICS OF VARIABLE MASS ROCKETS
HAVING LARGE AXIAL THRUST/ACCELERATION

A. JOSHI

Journal of Sound and Vibration (1995) **187**(4), 727–736

The free-free boundary conditions are:

$$w_1''(0)-a_1w_1(0)=0, \qquad w_1'''(0)-a_1w_1'(0)=0,$$

$$w_N''(\bar{l}_i)-a_1w_N(\bar{l}_i)=0, \qquad w_N'''(\bar{l}_i)-a_1w_N'(\bar{l}_i)=0,$$

and the continuity conditions are

$$w_i(\bar{l}_i)=w_j(0), \qquad w_i'(\bar{l}_i)=w_j'(0),$$

$$w_i''(\bar{l}_i)-a_iw_i(\bar{l}_i)=w_j''(0)-a_jw_j(0),$$

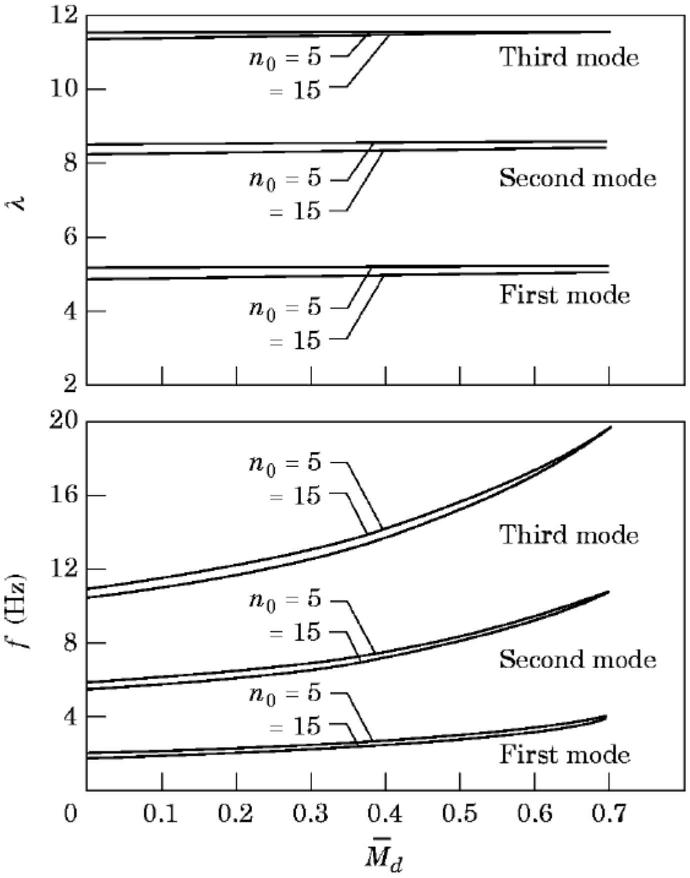
$$w_i'''(\bar{l}_i)-a_iw_i'(\bar{l}_i)=w_j'''(0)-a_jw_j'(0),$$

FREE VIBRATION CHARACTERISTICS OF VARIABLE MASS ROCKETS HAVING LARGE AXIAL THRUST/ACCELERATION

A. JOSHI

Journal of Sound and Vibration (1995) **187**(4), 727–736

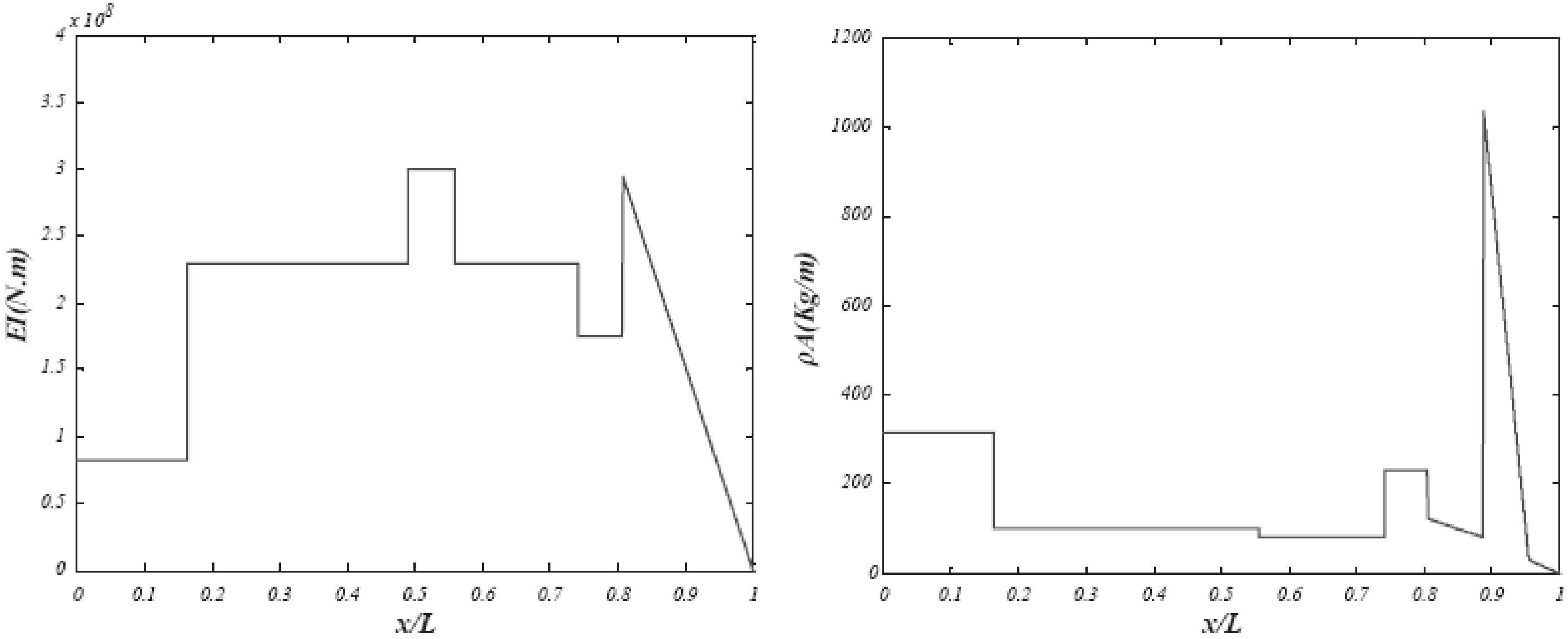
The variation of frequency parameter and cyclic frequency versus the mass depletion parameter M_d for the first three modes of vibration of a typical rocket executing a constant acceleration trajectory.



Investigation of thrust effect on the vibrational characteristics of flexible guided missiles

S.H. Pourtakdoust*, N. Assadian

Journal of Sound and Vibration 272 (2004) 287–299



(a) Distribution of bending stiffness at final time of flight; (b) distribution of mass density at final time of flight.

Investigation of thrust effect on the vibrational characteristics of flexible guided missiles

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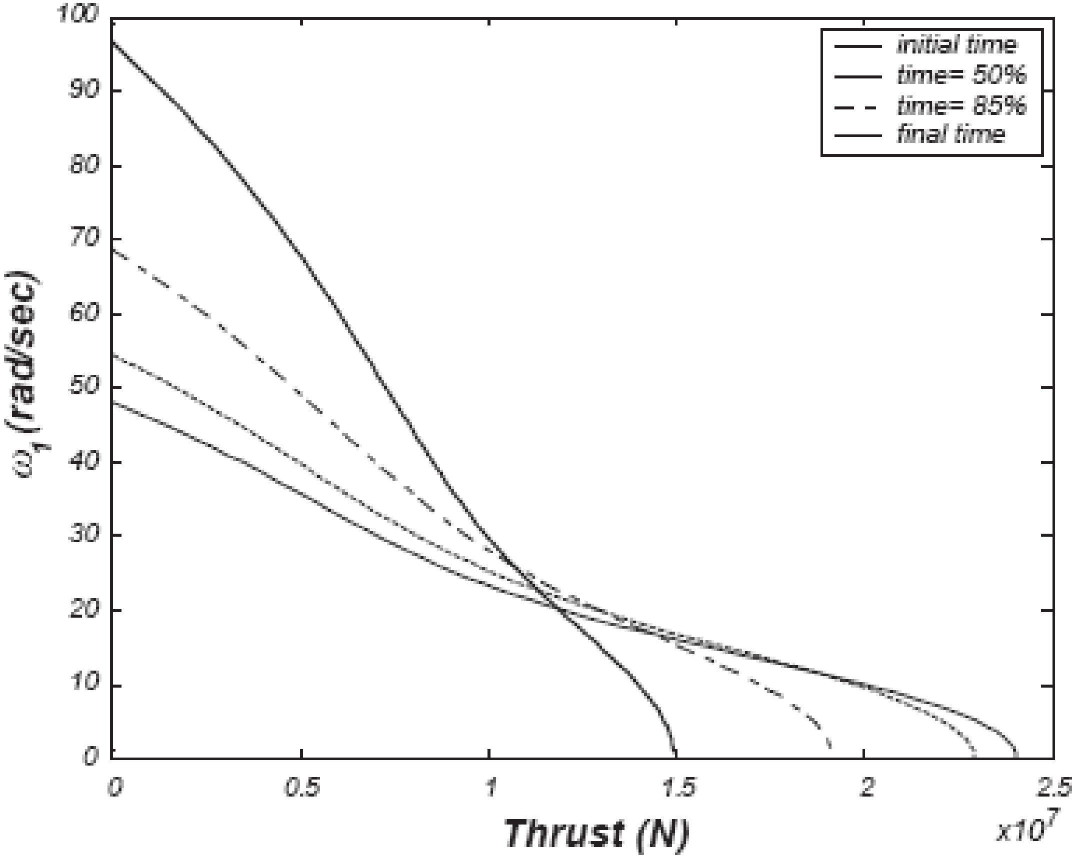

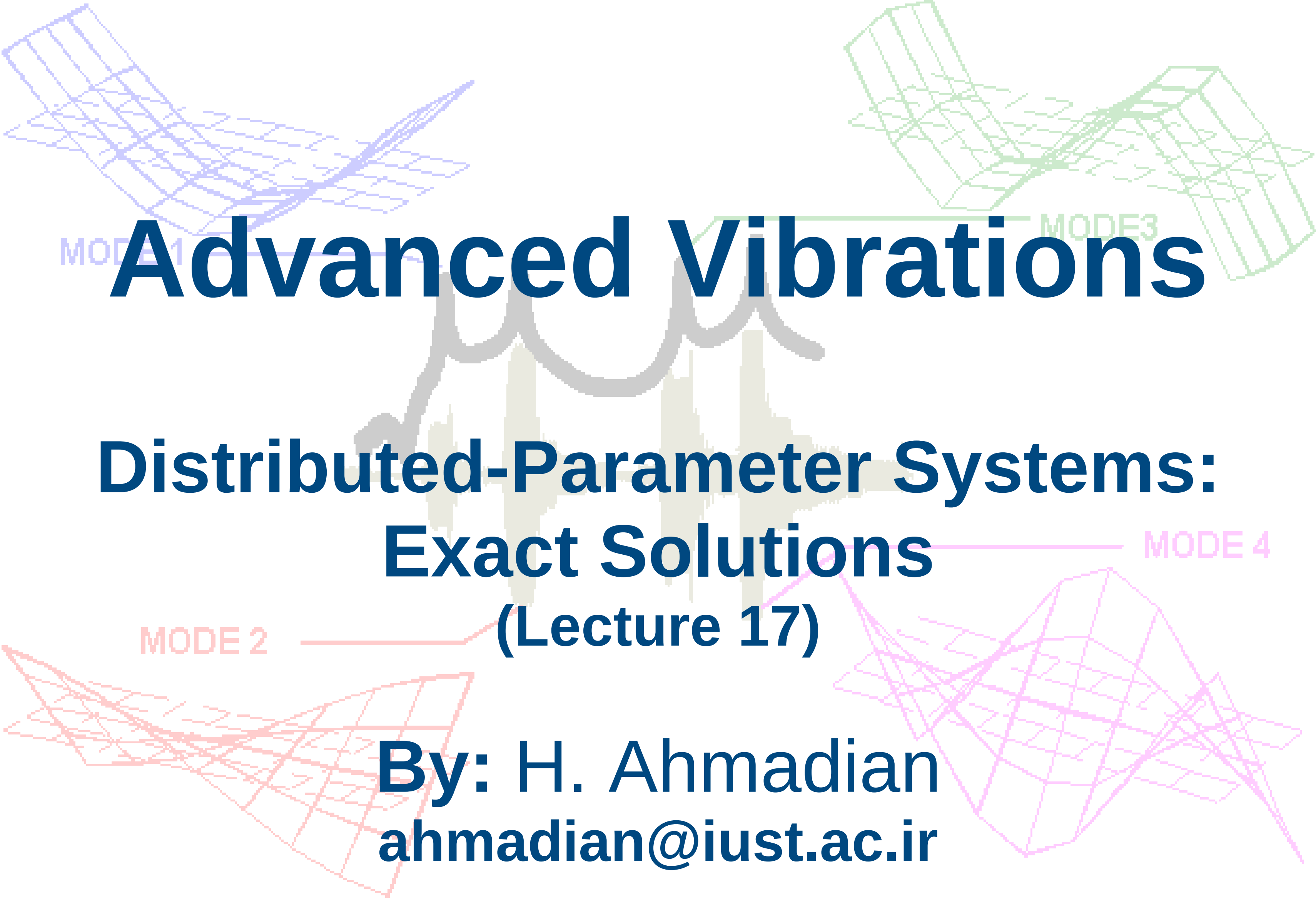

Advanced Vibrations

Distributed-Parameter Systems: Exact Solutions

(Lecture 17)

By: H. Ahmadian

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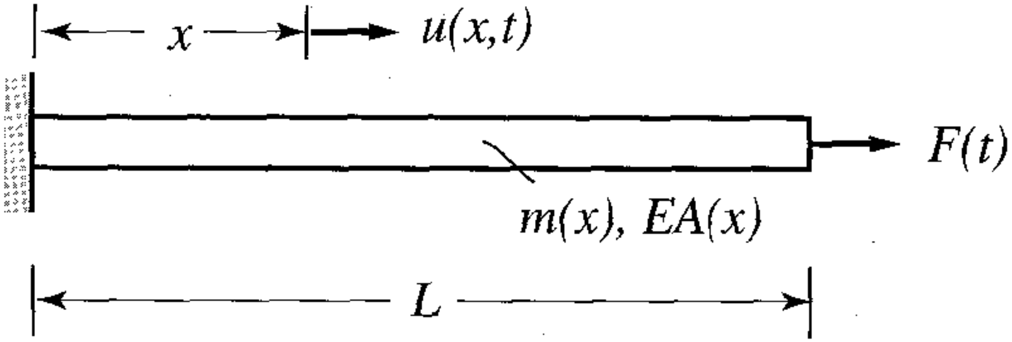
UMASS LOWELL MODAL ANALYSIS and CONTROLS LABORATORY - Pete Avitabile and Fabio Piergentili

SYSTEMS WITH EXTERNAL FORCES AT BOUNDARIES



$$u(0,t)=0,$$

$$EA(x)\frac{\partial u(x,t)}{\partial x}\bigg|_{x=L}=F(t)$$

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]=m(x)\frac{\partial^2 u(x,t)}{\partial t^2},\quad 0<x<L$$

The 2nd of boundary conditions is nonhomogeneous,

- precludes the use of modal analysis for the response.

SYSTEMS WITH EXTERNAL FORCES AT BOUNDARIES

We can reformulate the problem by rewriting the differential equation in the form:

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]+F(t)\delta(x-L)=m(x)\frac{\partial^2 u(x,t)}{\partial t^2},\quad 0<x<L$$

and the boundary conditions as:

$$u(0,t)=0,\quad \left[EA(x)\frac{\partial u(x,t)}{\partial x}\right]\Bigg|_{x=L}=0$$

Now the solution can be obtained routinely by modal analysis.

Any shortcomings?

SYSTEMS WITH EXTERNAL FORCES AT BOUNDARIES: Example

Obtain the response of a uniform rod, fixed at *x=0* and subjected to a boundary force at *x=L* in the form:

$$F(t) = F_0 u(t)$$

$$\frac{d^2U(x)}{dx^2} + \beta^2 U(x) = 0,\ 0 < x < L,\ \beta^2 = \frac{\omega^2 m}{EA}$$

$$U(0) = 0,\ \left.\frac{dU(x)}{dx}\right|_{x=L} = 0$$

$$\omega_r = \frac{(2r-1)\pi}{2}\sqrt{\frac{EA}{mL^2}},\quad U_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{(2r-1)\pi x}{2L},\ r = 1, 2, \ldots$$

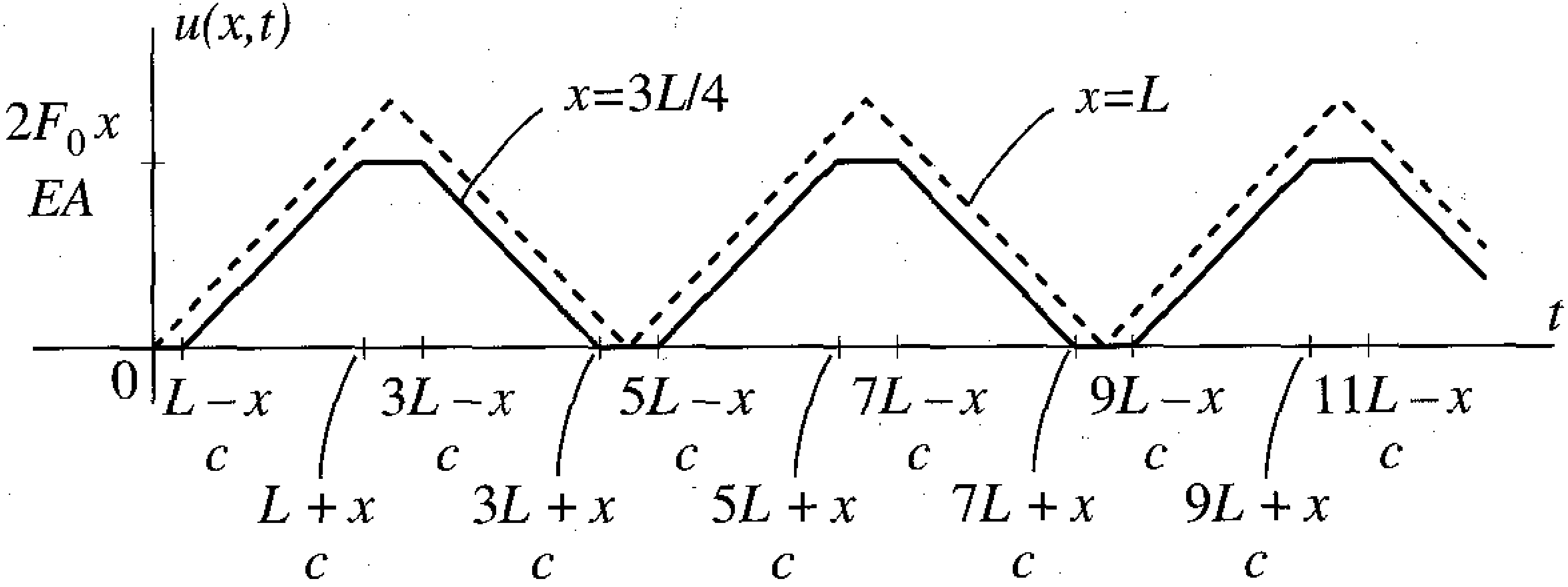
SYSTEMS WITH EXTERNAL FORCES AT BOUNDARIES: Example

$$\eta_r(t) = \frac{U_r(L)}{\omega_r} \int_0^t F_0 \mathscr{u}(t-\tau) \sin \omega_r \tau \, d\tau = \frac{F_0 U_r(L)}{\omega_r^2} (1 - \cos \omega_r t)$$

$$= \frac{4F_0 \sqrt{2/mL} \sin \dfrac{(2r-1)\pi}{2}}{(2r-1)^2 \pi^2} \frac{mL^2}{EA} \left[1 - \cos \frac{(2r-1)\pi}{2} \sqrt{\frac{EA}{mL^2}} t \right]$$

$$= \frac{4F_0 \sqrt{2/mL} (-1)^{r-1}}{(2r-1)^2 \pi^2} \frac{mL^2}{EA} \left[1 - \cos \frac{(2r-1)\pi}{2} \sqrt{\frac{EA}{mL^2}} t \right], \quad r = 1, 2, \ldots$$

$$u(x,t) = \frac{8F_0 L}{\pi^2 EA} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^2} \sin \frac{(2r-1)\pi x}{2L} \left[1 - \cos \frac{(2r-1)\pi}{2} \sqrt{\frac{EA}{mL^2}} t \right]$$

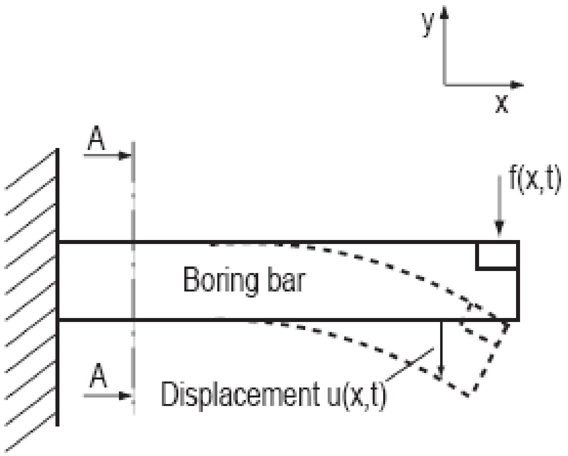


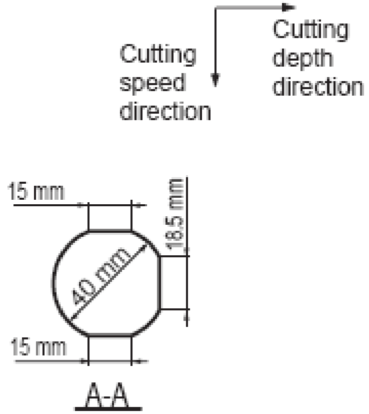
Axial displacement at $x = 3L/4$ due to a force in in the form of a step function at $x = L$

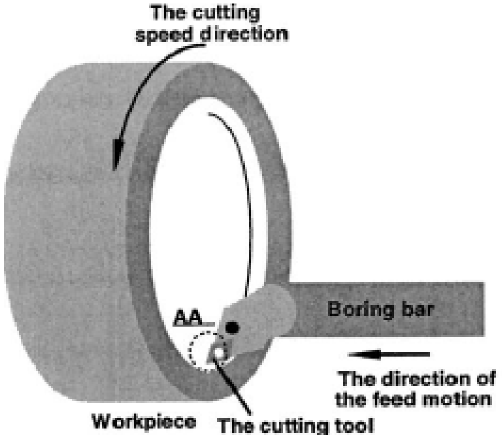
Identification of dynamic properties of boring bar vibrations in a continuous boring operation

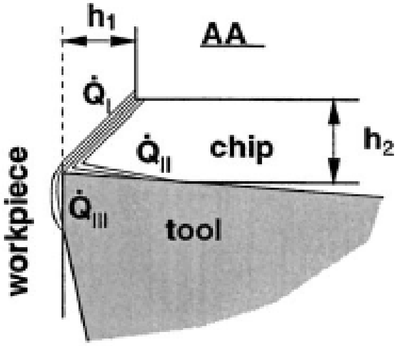
L. Andrén*, L. Håkansson, A. Brandt, I. Claesson

Mechanical Systems and Signal Processing 18 (2004) 869–901









$$\rho A(x)\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right]=f(x,t),$$

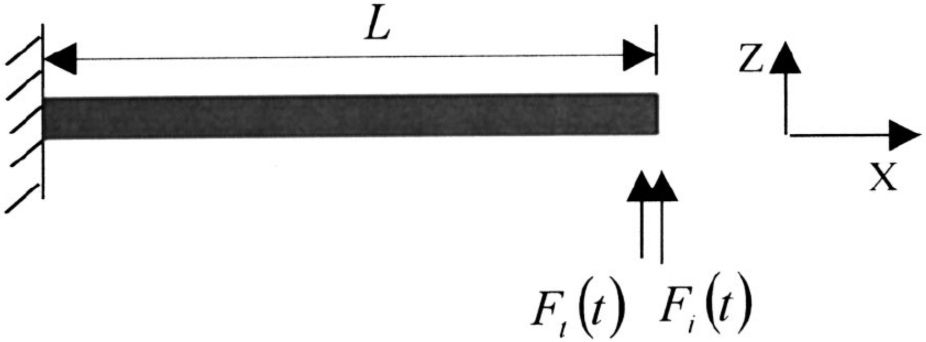
$$u(x,t)\big|_{x=0}=0,\quad \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}=0.\qquad \left.EI(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{x=l}=0,\quad \left.\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 u(x,t)}{\partial x^2}\right)\right|_{x=l}=0.$$

Flank Wear and Process Characteristic Effect on System Dynamics in Turning

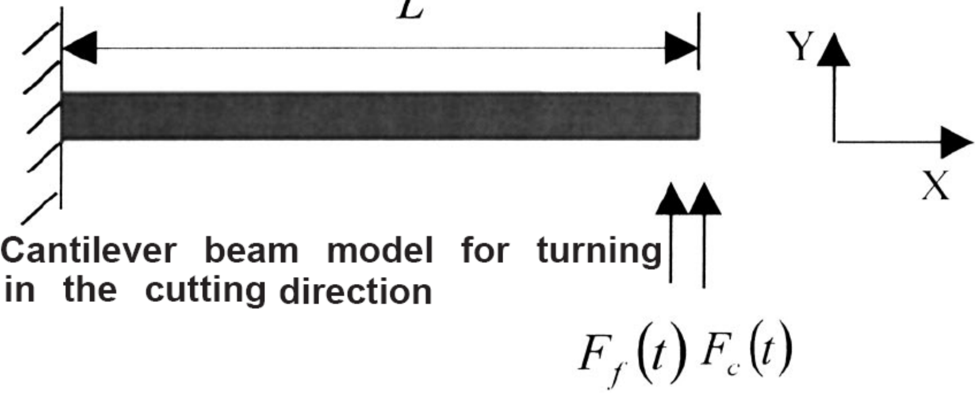
Ming-Chyuan Lu
Graduate Student

Elijah Kannatey-Asibu, Jr.
Professor

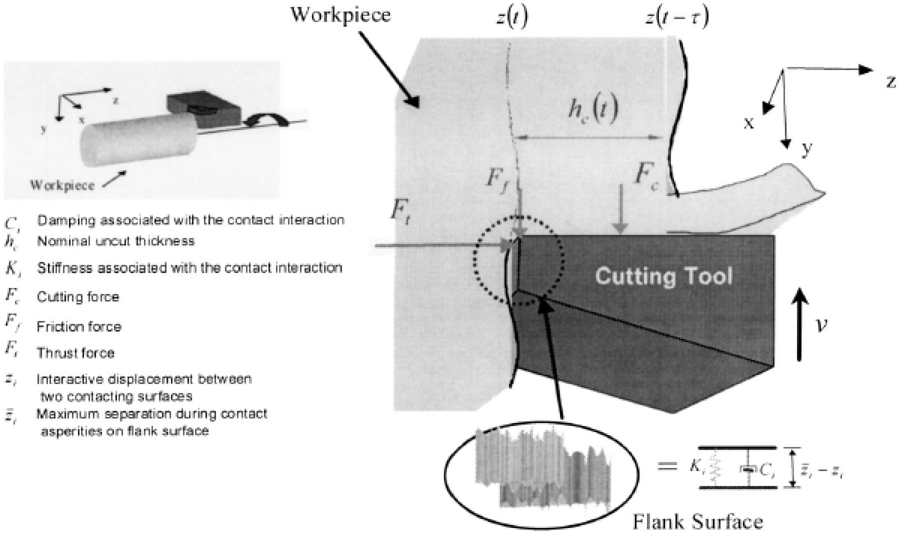
Journal of Manufacturing Science and Engineering
FEBRUARY 2004, Vol. 126



Cantilever beam model for turning in the feed direction



Cantilever beam model for turning in the cutting direction



C_i	Damping associated with the contact interaction
h_c	Nominal uncut thickness
K_i	Stiffness associated with the contact interaction
F_c	Cutting force
F_f	Friction force
F_t	Thrust force
z_i	Interactive displacement between two contacting surfaces
$\bar{z}_i$	Maximum separation during contact asperities on flank surface

System Dynamics in the Feed Direction

$$EI\frac{\partial z^{4}(x,t)}{\partial x^{4}}+\rho A\frac{\partial z^{2}(x,t)}{\partial t^{2}}=0 \qquad (0<x<L)$$

$$z(x,t)\big|_{x=0}=0, \qquad \frac{\partial z(x,t)}{dx}\bigg|_{x=0}=0. \qquad \frac{d^{2}z(x,t)}{dx^{2}}\bigg|_{x=L}=0$$

$$EI\frac{d^{3}z(x,t)}{dx^{3}}\bigg|_{x=L}=F_{t}(t)+F_{i}(t)$$

$$F_{t}(t) = k_{zq}[z(L,t)-z(L,t-\tau)+h_{c}]$$

$$F_{i}(t) = K_{i}z_{i}(t)+C_{i}\dot{z}_{i}(t)$$

System Dynamics in the Cutting Direction

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+\rho A\frac{\partial^2 y(x,t)}{\partial t^2}=0$$

$$y(0,t)=0 \quad \left.\frac{\partial y(x,t)}{dx}\right|_{x=0}=0. \quad \left.\frac{d^2 y(x,t)}{dx^2}\right|_{x=L}=0$$

$$EI\left.\frac{d^3 y(x,t)}{dx^3}\right|_{x=L}=F_c(t)+F_f(t).$$

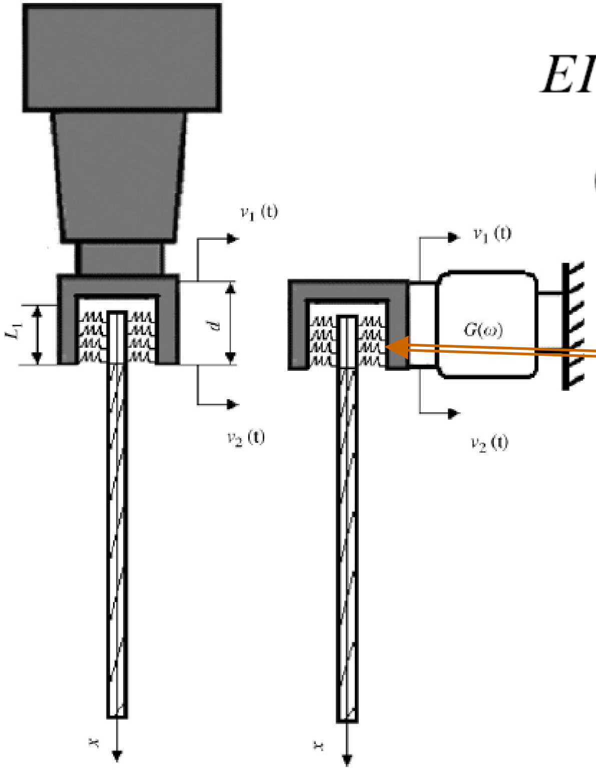
$$F_c(t)=k_{yq}[z(L,t)-z(L,t-\tau)+h_c]$$

$$F_f(t)=\mu_d[F_i(t)+F_t(t)]$$

Modelling machine tool dynamics using a distributed parameter tool–holder joint interface

Keivan Ahmadi, Hamid Ahmadian*

International Journal of Machine Tools & Manufacture 47 (2007) 1916–1928



$$EI_1 \frac{\partial^4 U_1(x,t)}{\partial x^4} + m_1 \frac{\partial^2 U_1(x,t)}{\partial t^2} = K(x)[v(x,t) - U_1(x,t)],$$

$$0 \leqslant x \leqslant L_1,$$

$$K(x) = k(x)(1 + \mathrm{i}\eta),$$

$$EI_2 \frac{\partial^4 U_2(x,t)}{\partial x^4} + m_2 \frac{\partial^2 U_2(x,t)}{\partial t^2} = 0, \quad L_1 \leqslant x \leqslant L,$$

Modeling Tool as Stepped Beam on Elastic Support: Boundary Conditions

$$\frac{\partial^2 U_1(0,t)}{\partial x^2} = 0,$$

$$\frac{\partial^3 U_1(0,t)}{\partial x^3} = 0.$$

$$-EI_2 \frac{\partial^3 U_2(L,t)}{\partial x^3} = \mathrm{e}^{\mathrm{i}\omega t},$$

$$\frac{\partial^2 U_2(L,t)}{\partial x^2} = 0.$$

Modeling Tool as Stepped Beam on Elastic Support: The compatibility requirements

$$U_1(L_1,t) - U_2(L_1,t) = 0,$$

$$\frac{\partial U_1(L_1,t)}{\partial x} - \frac{\partial U_2(L_1,t)}{\partial x} = 0,$$

$$EI_1 \frac{\partial^2 U_1(L_1,t)}{\partial x^2} - EI_2 \frac{\partial^2 U_2(L_1,t)}{\partial x^2} = 0,$$

$$EI_1 \frac{\partial^3 U_1(L_1,t)}{\partial x^3} - EI_2 \frac{\partial^3 U_2(L_1,t)}{\partial x^3} = 0.$$

Modeling Tool as Stepped Beam on Elastic Support

$$U_1(x,t) = \Phi(x)\mathrm{e}^{\mathrm{i}\omega t},$$

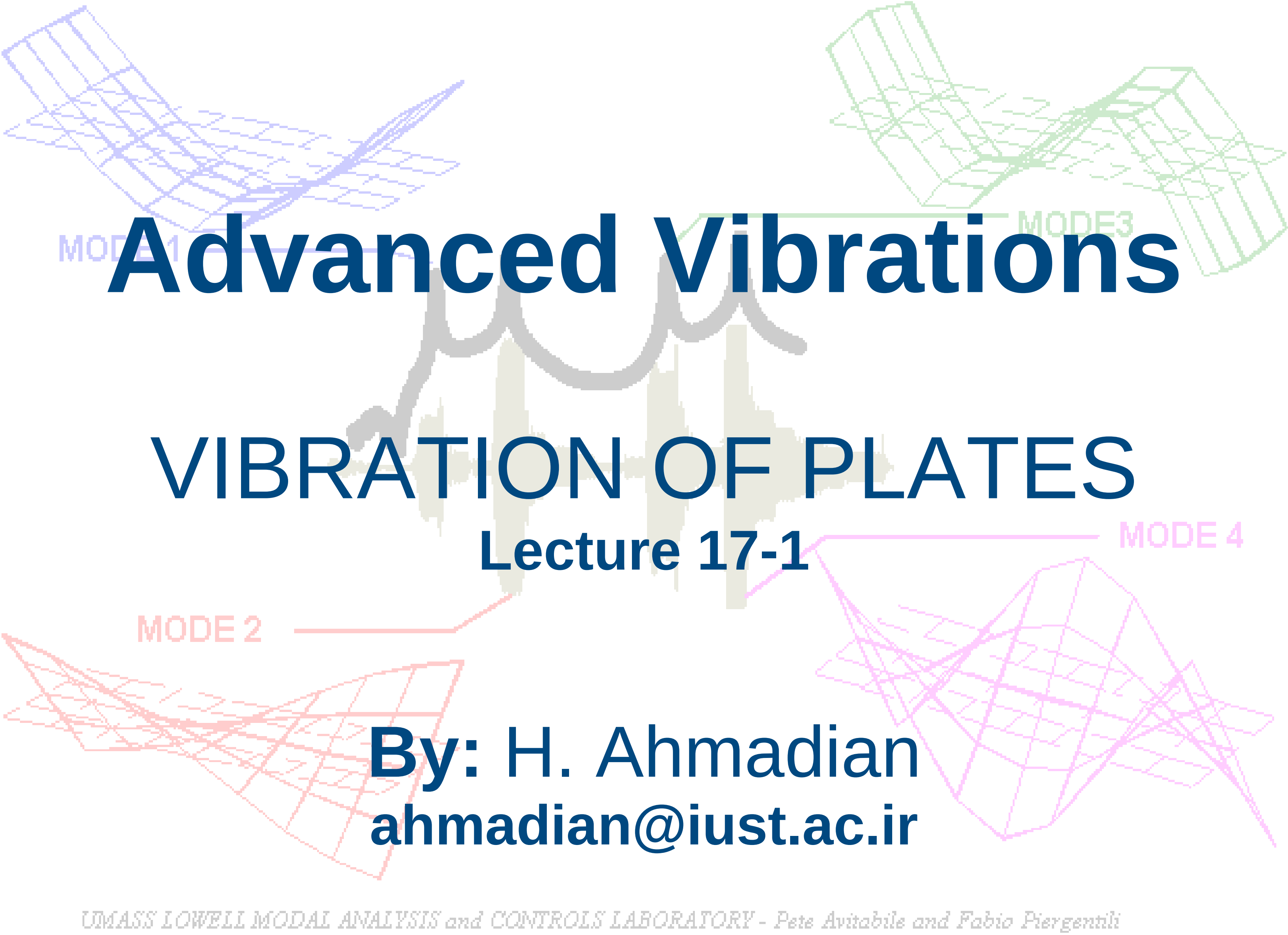
$$U_2(x,t) = \Psi(x)\mathrm{e}^{\mathrm{i}\omega t},$$

$$K(x) = \sum_{p=0}^{P} K_p x^p \Longrightarrow \Phi(x) = \sum_{n=1}^{N} a_n x^{n-1}$$

$$\Psi(x) = C_1\mathrm{e}^{\mathrm{i}\lambda x} + C_2\mathrm{e}^{-\mathrm{i}\lambda x} + C_3\mathrm{e}^{\lambda x} + C_4\mathrm{e}^{-\lambda x},$$

Modeling Tool as Stepped Beam on Elastic Support

$$[Z(\omega)]\begin{Bmatrix} C_1 \\ \vdots \\ C_4 \\ a_1 \\ \vdots \\ a_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{Bmatrix},$$



Advanced Vibrations

VIBRATION OF PLATES

Lecture 17-1

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VIBRATION OF PLATES

- Plates have bending stiffness in a manner similar to beams in bending.
- In the case of plates one can think of two planes of bending, producing in general two distinct curvatures.
- The small deflection theory of thin plates, called *classical plate theory* or *Kirchhoff theory,* is based on assumptions similar to those used in thin beam or Euler-Bernoulli beam theory.

EQUATION OF MOTION: CLASSICAL PLATE THEORY

The *elementary theory of plates* is based on the following assumptions:

- The thickness of the plate (h) is small compared to its lateral dimensions.
- The middle plane of the plate does not undergo in-plane deformation. Thus, the midplane remains as the neutral plane after deformation or bending.
- The displacement components of the midsurface of the plate are small compared to the thickness of the plate.
- The influence of transverse shear deformation is neglected. This implies that plane sections normal to the midsurface before deformation remain normal to the rnidsurface even after deformation or bending.
- The transverse normal strain under transverse loading can be neglected. The transverse normal stress is small and hence can be neglected compared to the other components of stress.

Moment - Shear Force Resultants:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

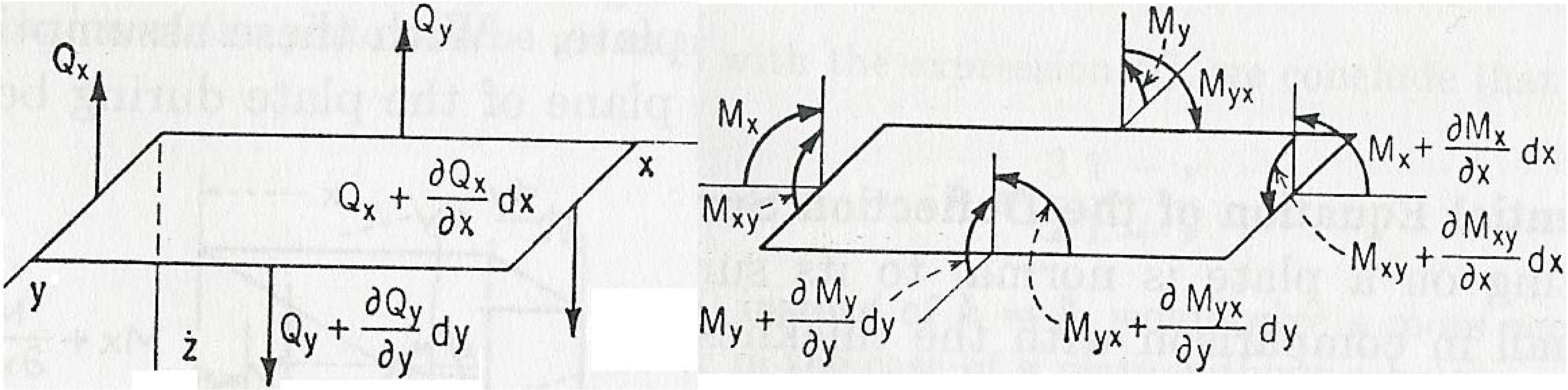
$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \qquad D = \frac{Eh^3}{12(1-\nu^2)}$$

$$M_{xy} = M_{yx} = -(1-\nu)D\frac{\partial^2 w}{\partial x\,\partial y}$$

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

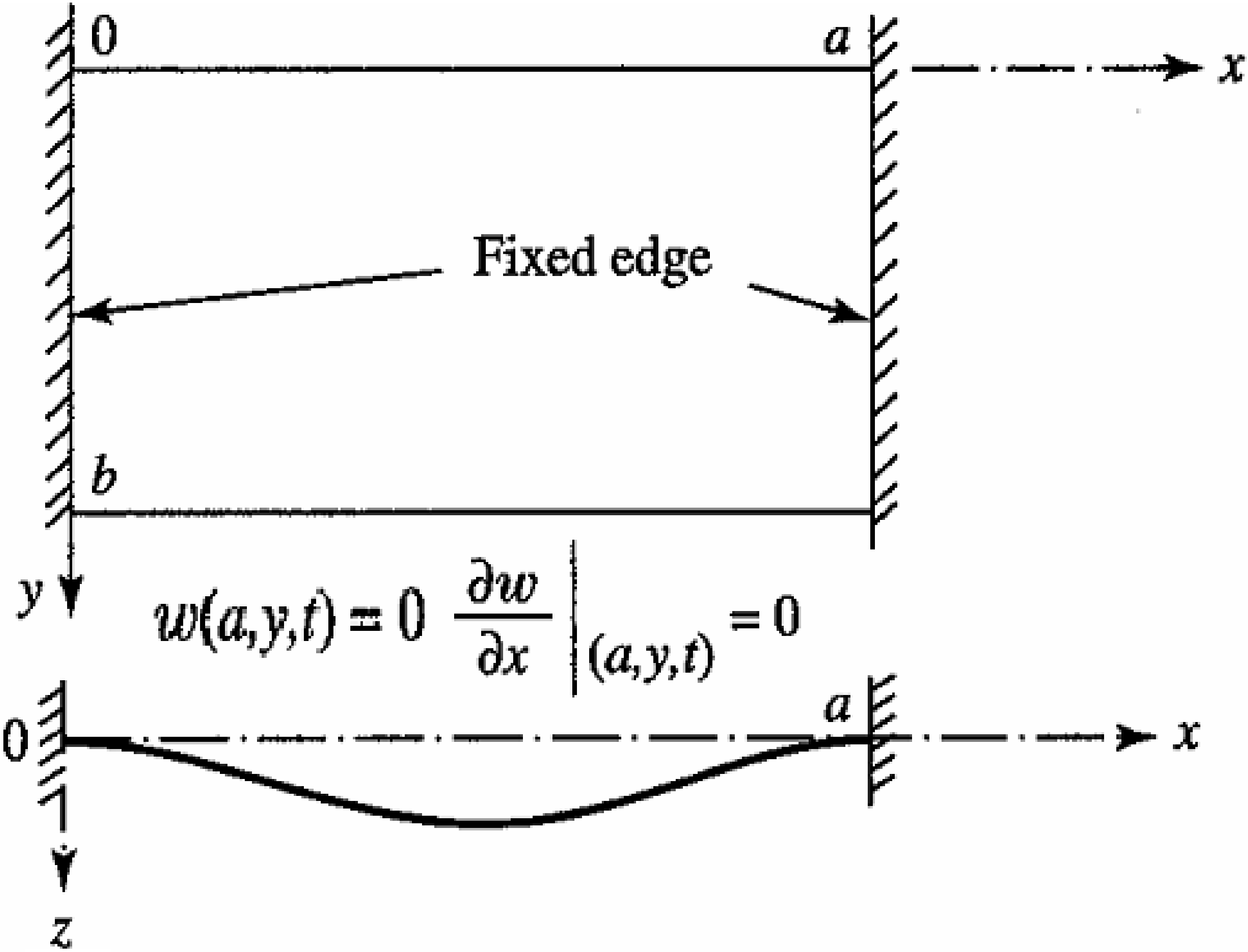
Equation of motion



$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + f(x,y,t) = \rho h \frac{\partial^2 w}{\partial t^2}$$

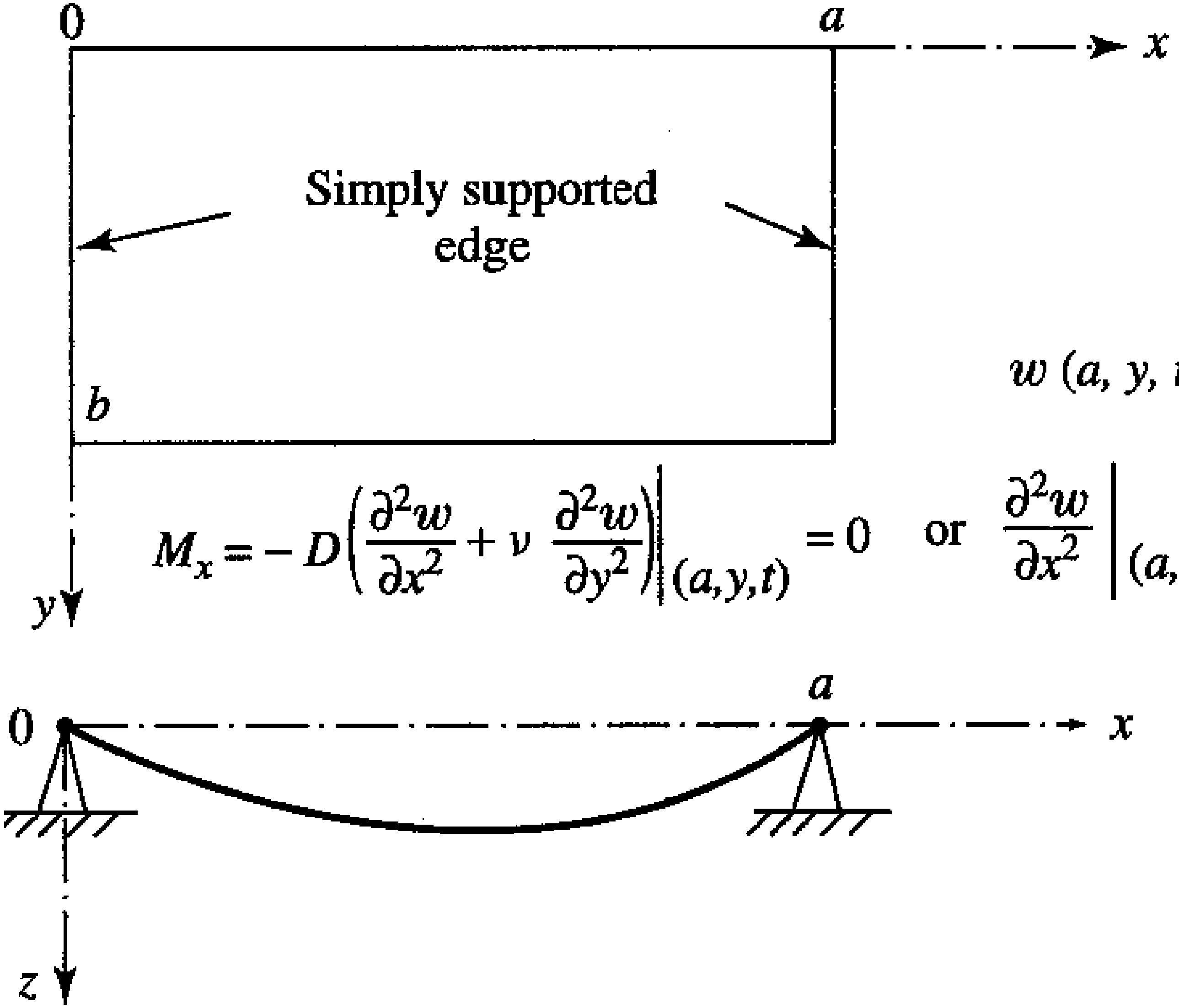
$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) + \rho h \frac{\partial^2 w}{\partial t^2} = f(x,y,t)$$

BOUNDARY CONDITIONS



$$w(a,y,t) = 0 \quad \left.\frac{\partial w}{\partial x}\right|_{(a,y,t)} = 0$$

BOUNDARY CONDITIONS



$$w\,(a,\,y,\,t)=0$$

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\,\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{(a,y,t)}=0 \quad \text{or} \quad \frac{\partial^2 w}{\partial x^2}\Bigg|_{(a,y,t)}=0$$

BOUNDARY CONDITIONS: Free Edge

- There are three boundary conditions, whereas the equation of motion requires only two:

$$M_x\big|_{x=a} = 0 \quad Q_x\big|_{x=a} = 0 \quad M_{xy}\big|_{x=a} = 0$$

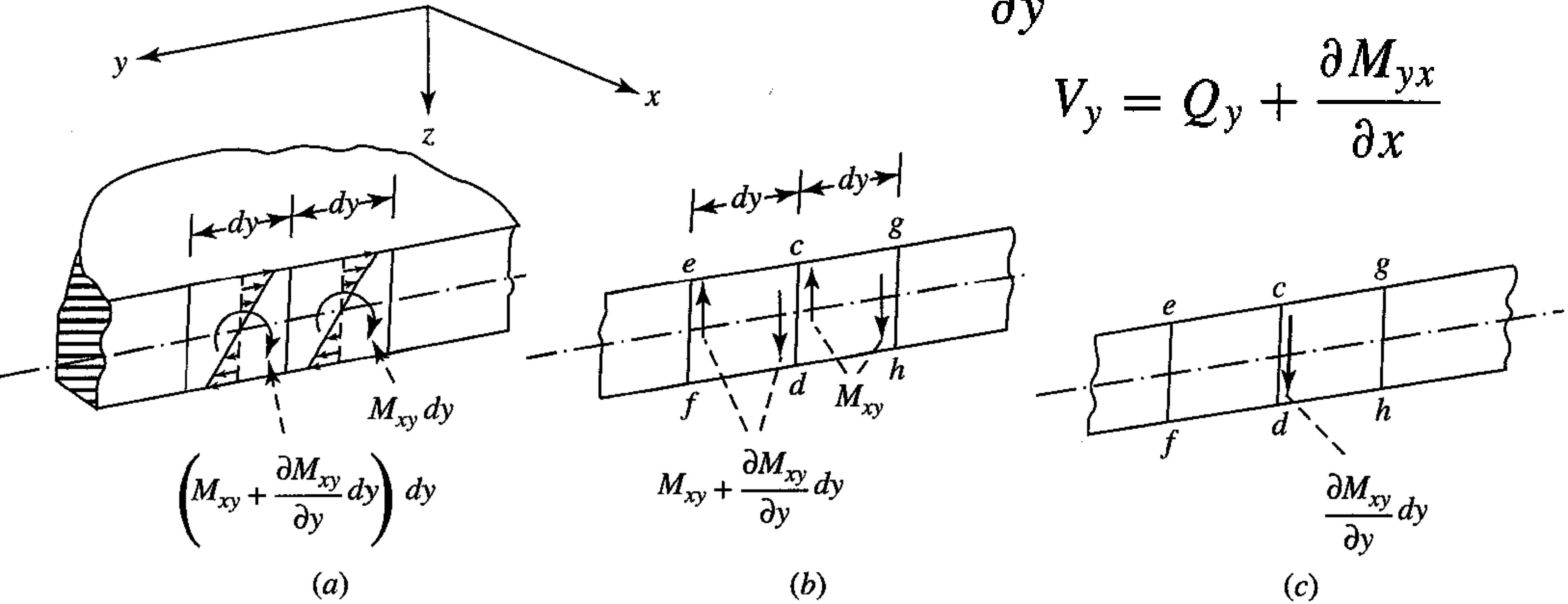
- Kirchhoff showed that the conditions on the shear force and the twisting moment are not independent and can be combined into only one boundary condition.

BOUNDARY CONDITIONS: Free Edge

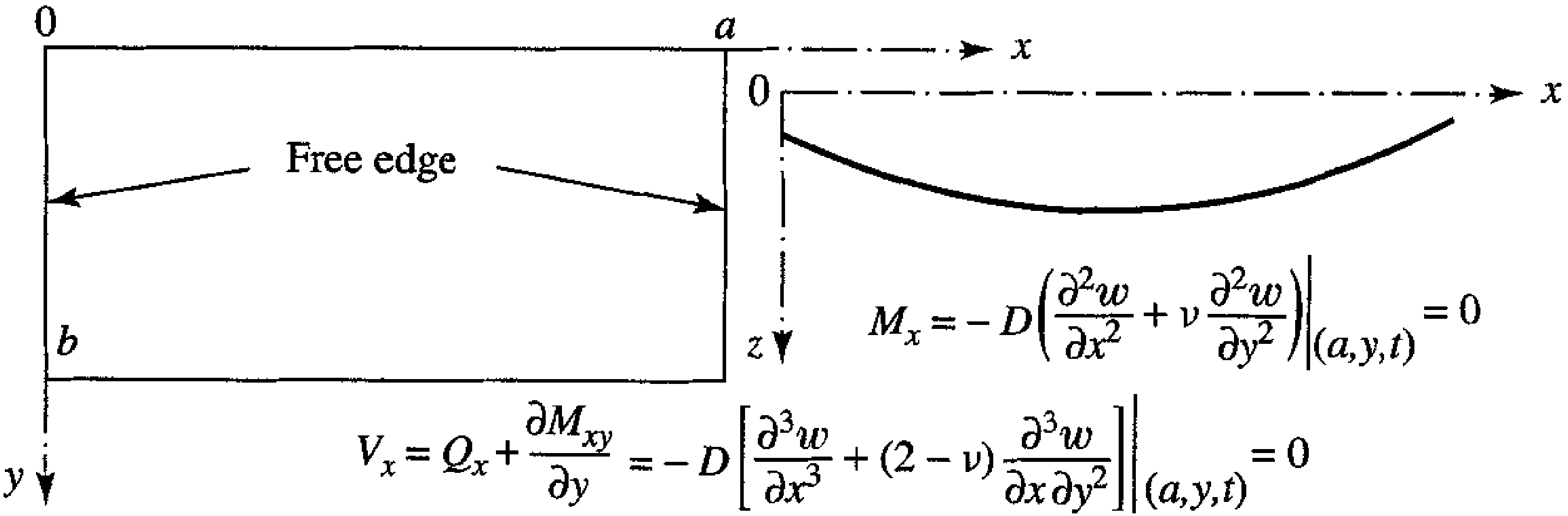
Replacing the twisting moment by an equivalent vertical force.

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y}$$

$$V_y = Q_y + \frac{\partial M_{yx}}{\partial x}$$



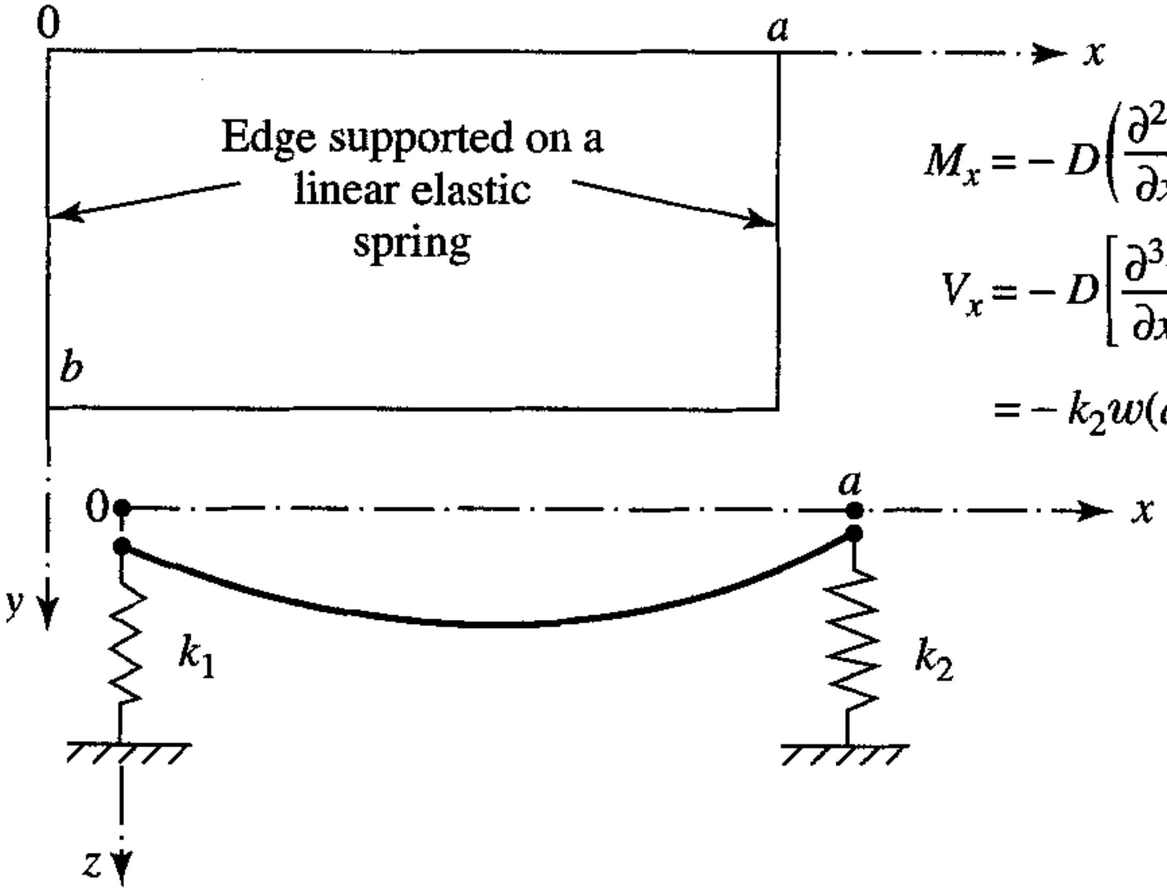
BOUNDARY CONDITIONS



$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{(a,y,t)} = 0$$

$$V_x = Q_x + \frac{\partial M_{xy}}{\partial y} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\, \partial y^2}\right]\Bigg|_{(a,y,t)} = 0$$

BOUNDARY CONDITIONS

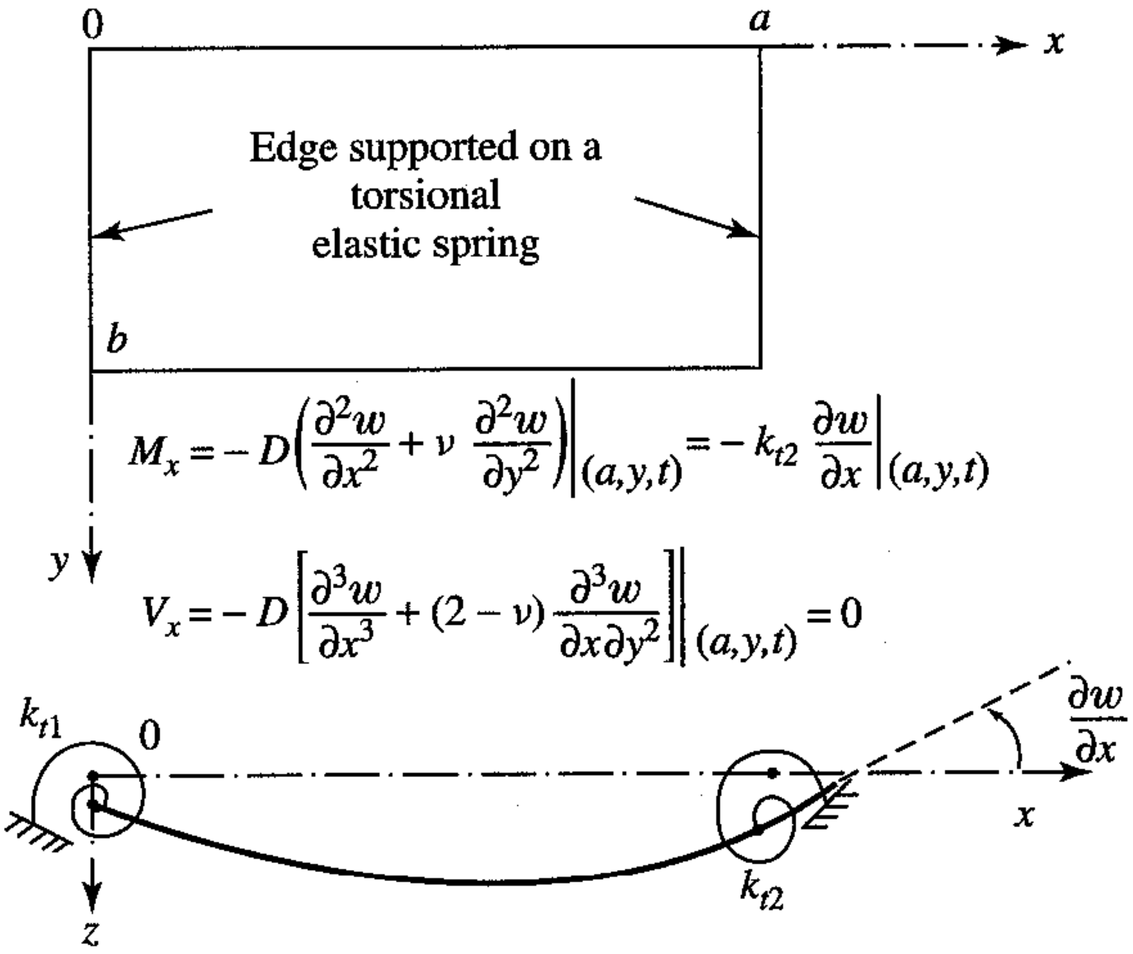


$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{(a,y,t)} = 0$$

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]\Bigg|_{(a,y,t)}$$

$$= -k_2 w(a,y,t)$$

BOUNDARY CONDITIONS



$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\bigg|_{(a,y,t)} = -k_{t2}\frac{\partial w}{\partial x}\bigg|_{(a,y,t)}$$

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]\bigg|_{(a,y,t)} = 0$$

FREE VIBRATION OF RECTANGULAR PLATES

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0$$

$$w(x,y,t)=W(x,y)T(t)$$

$$\frac{d^2T(t)}{dt^2}+\omega^2T(t)=0 \quad T(t)=A\cos\omega t+B\sin\omega t$$

$$\nabla^4W(x,y)-\lambda^4W(x,y)=0 \qquad \lambda^4=\frac{\rho h\omega^2}{D}$$

FREE VIBRATION OF RECTANGULAR PLATES

$$(\nabla^4 - \lambda^4)W(x,y) = (\nabla^2 + \lambda^2)(\nabla^2 - \lambda^2)W(x,y) = 0$$

$$(\nabla^2 + \lambda^2)W_1(x,y) = \frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2} + \lambda^2 W_1(x,y) = 0$$

$$(\nabla^2 - \lambda^2)W_2(x,y) = \frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_2}{\partial y^2} - \lambda^2 W_2(x,y) = 0$$

FREE VIBRATION OF RECTANGULAR PLATES

$$\begin{aligned} W(x,y) = {} & A_1 \sin \alpha x \sin \beta y + A_2 \sin \alpha x \cos \beta y \\ & + A_3 \cos \alpha x \sin \beta y + A_4 \cos \alpha x \cos \beta y \\ & + A_5 \sinh \theta x \sinh \phi y + A_6 \sinh \theta x \cosh \phi y \\ & + A_7 \cosh \theta x \sinh \phi y + A_8 \cosh \theta x \cosh \phi y \end{aligned}$$

$$\lambda^2 = \alpha^2 + \beta^2 = \theta^2 + \phi^2$$

Solution for a Simply Supported Plate

$$W(0, y) = \frac{d^2W}{dx^2}(0, y) = W(a, y) = \frac{d^2W}{dx^2}(a, y) = 0$$

$$W(x, 0) = \frac{d^2W}{dy^2}(x, 0) = W(x, b) = \frac{d^2W}{dy^2}(x, b) = 0$$

We find that all the constants A_i except A_1 *and*

$$\sin \alpha a = 0 \longrightarrow \alpha_m a = m\pi, \qquad m = 1, 2, \ldots$$

$$\sin \beta b = 0 \longrightarrow \beta_n b = n\pi, \qquad n = 1, 2, \ldots$$

$$\omega_{mn} = \lambda_{mn}^2 \left(\frac{D}{\rho h}\right)^{1/2} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \left(\frac{D}{\rho h}\right)^{1/2},$$

$$W_{mn}(x, y) = A_{1mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \qquad m, n = 1, 2, \ldots$$

Solution for a Simply Supported Plate

$$w_{mn}(x,y,t) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t)$$

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t)$$

The initial conditions of the plate are:

$$w(x,y,0) = w_0(x,y)$$

$$\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$$

Solution for a Simply Supported Plate

$$w(x,y,0) = w_0(x,y)$$

$$\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$$

→

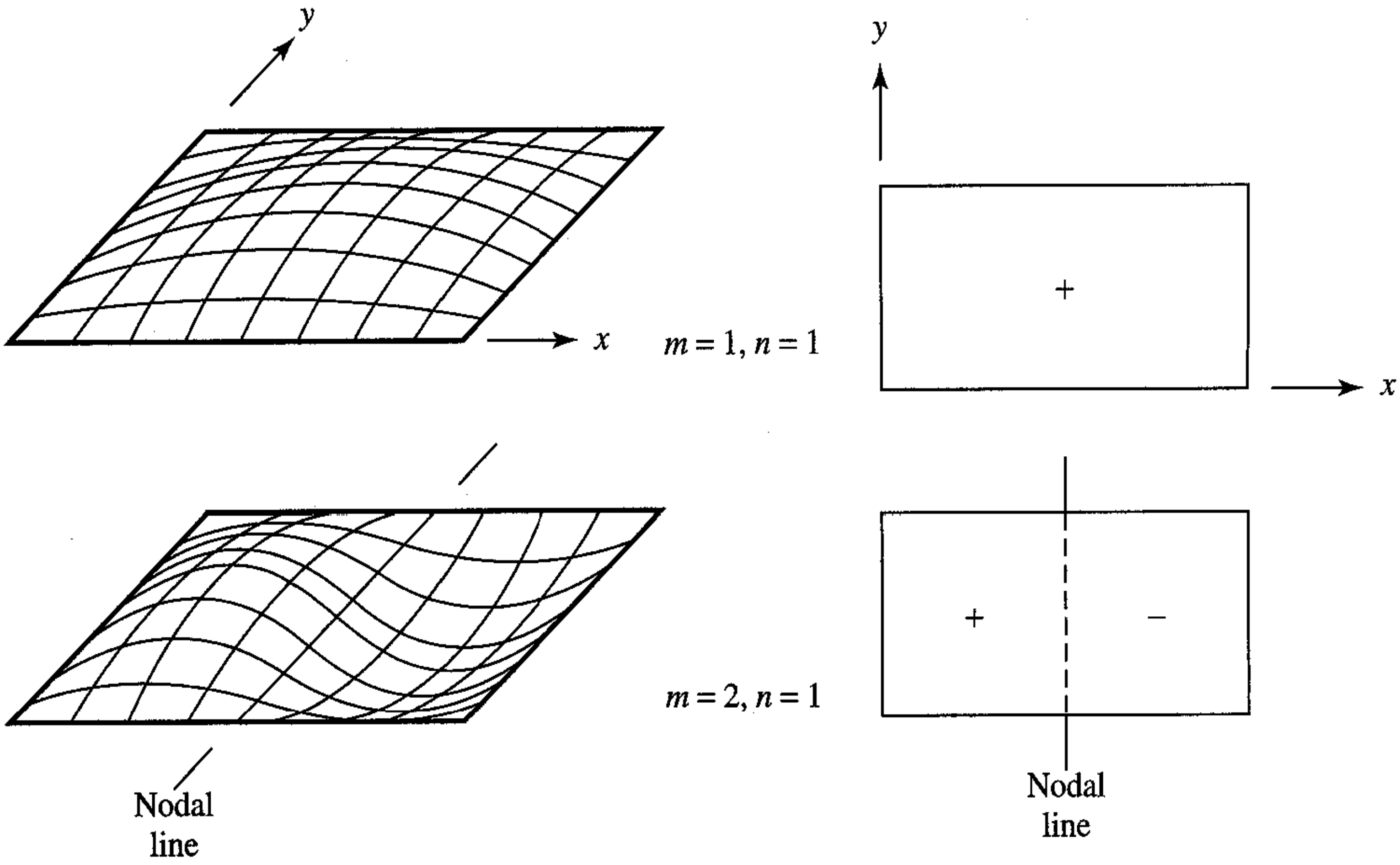
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = w_0(x,y)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\omega_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} = \dot{w}_0(x,y)$$

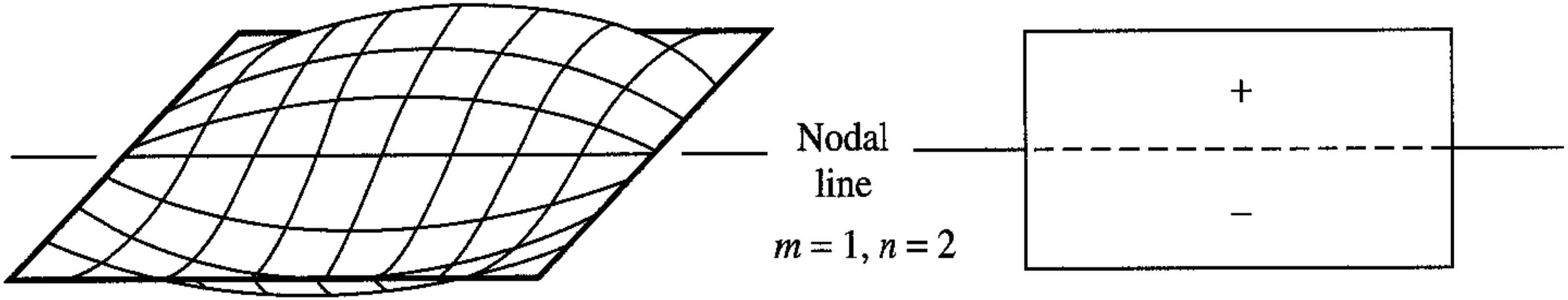
$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b w_0(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

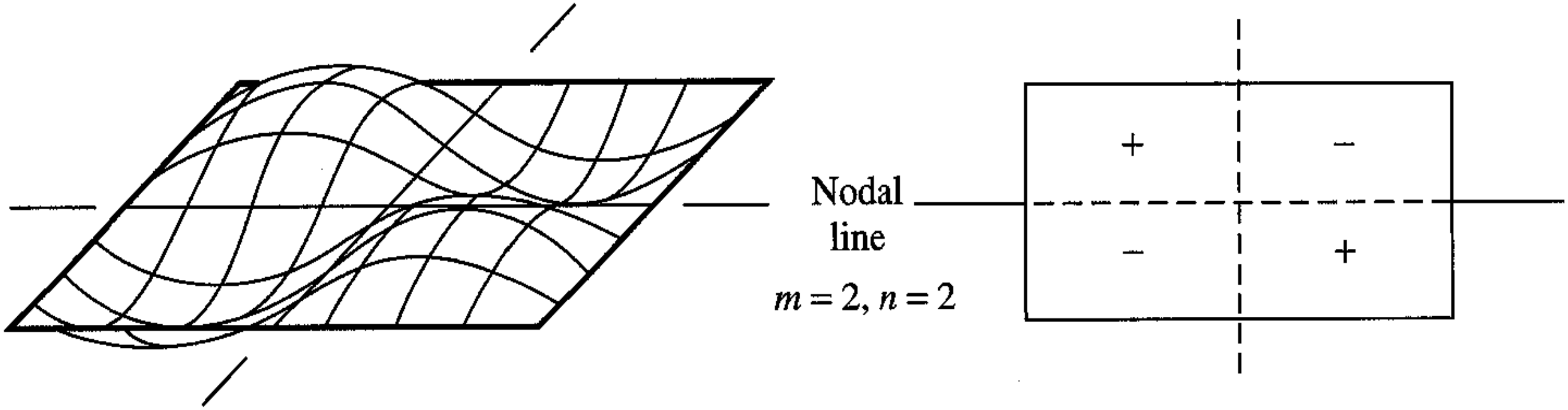
$$B_{mn} = \frac{4}{ab\omega_{mn}}\int_0^a\int_0^b \dot{w}_0(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

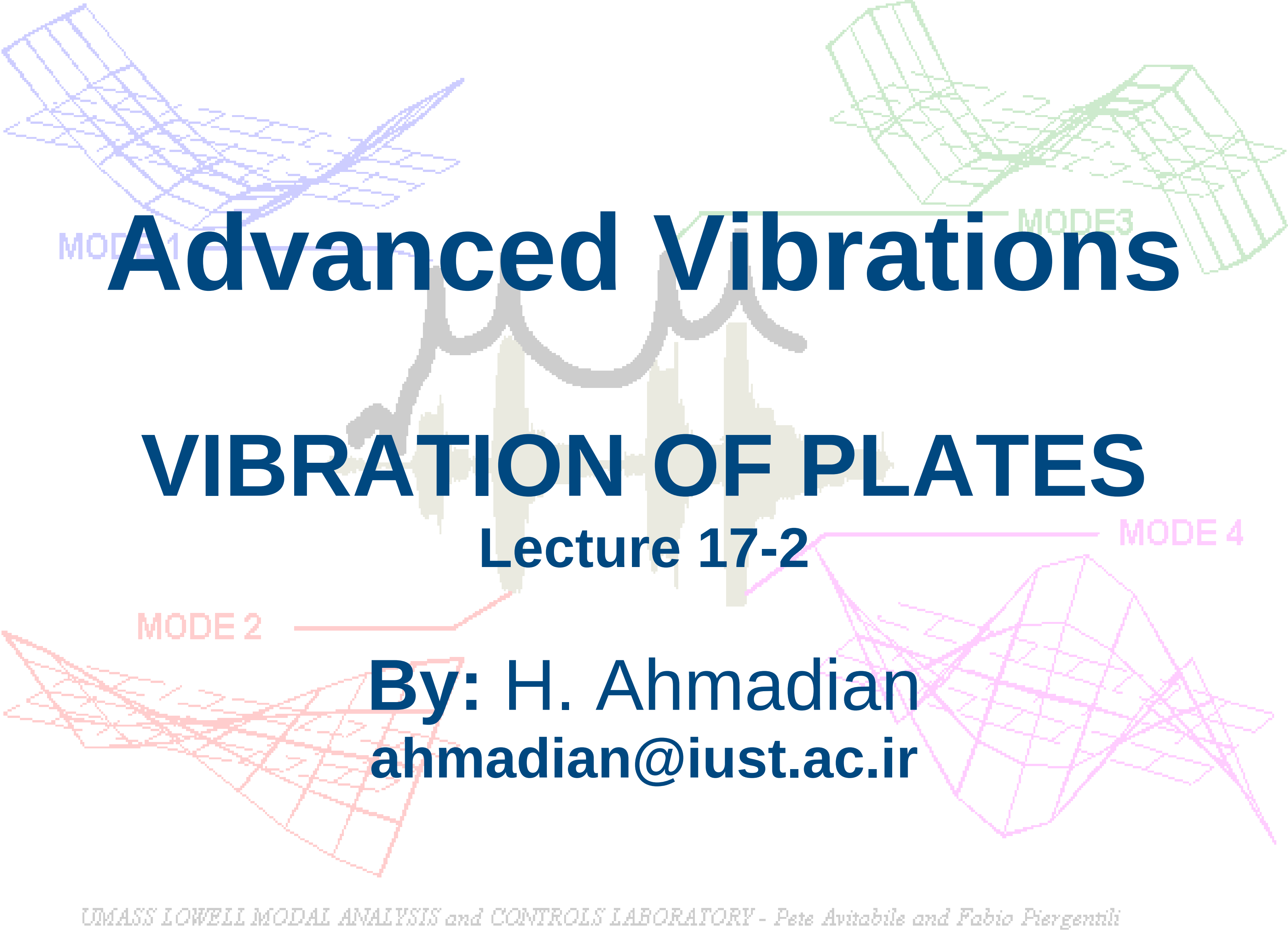
Solution for a Simply Supported Plate



Solution for a Simply Supported Plate







UMASS LOWELL MODAL ANALYSIS and CONTROLS LABORATORY - Pete Avitabile and Fabio Piergentili

Advanced Vibrations

VIBRATION OF PLATES

Lecture 17-2

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Vibrations of Rectangular Plates

$$\nabla^4 W(x, y) - \lambda^4 W(x, y) = 0 \quad \lambda^4 = \frac{\rho h \omega^2}{D}$$

$$W(x,y) = X(x)Y(y)$$

$$X'''' Y + 2X''Y'' + XY'''' - \lambda^4 XY = 0$$

The functions *X(x)* and *Y(y)* can be separated provided either of the followings are satisfied:

$$\Longrightarrow Y''(y) = -\beta^2 Y(y), Y''''(y) = -\beta^2 Y''(y)$$

$$\Longrightarrow X''(x) = -\alpha^2 X(x), X''''(x) = -\alpha^2 X''(x)$$

Vibrations of Rectangular Plates

$$Y''(y) = -\beta^2 Y(y), Y''''(y) = -\beta^2 Y''(y)$$
$$X''(x) = -\alpha^2 X(x), X''''(x) = -\alpha^2 X''(x)$$

These equations can be satisfied only by the trigonometric functions:

$$\begin{Bmatrix} \sin \alpha_m x \\ \cos \alpha_m x \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \sin \beta_n y \\ \cos \beta_n y \end{Bmatrix}$$

$$\alpha_m = \frac{m\pi}{a}, m = 1, 2, \ldots, \beta_n = \frac{n\pi}{b}, n = 1, 2, \ldots$$

Vibrations of Rectangular Plates

Assume that the plate is simply supported along edges *x =0* and *x =a*:

$$X_m(x) = A \sin \alpha_m x, \qquad m = 1, 2, \ldots$$

$$X_m(0) = X_m(a) = X''_m(0) = X''_m(a) = 0$$

Implying:

$$w(0,y,t) = w(a,y,t) = \nabla^2 w(0,y,t) = \nabla^2 w(a,y,t) = 0$$

$$Y''''(y) - 2\alpha_m^2 Y''(y) - (\lambda^4 - \alpha_m^4) Y(y) = 0$$

Vibrations of Rectangular Plates

The various boundary conditions can be stated,

SS-SS-SS-SS, SS-C-SS-C, SS-F-SS-F, SS-C-SS-SS, SS-F-SS-SS, SS-F-SS-C

Assuming: $\lambda^4 > \alpha_m^4$

$$Y(y) = e^{sy}$$

$$s^4 - 2s^2\alpha_m^2 - (\lambda^4 - \alpha_m^4) = 0$$

$$s_{1,2} = \pm\sqrt{\lambda^2 + \alpha_m^2}, \qquad s_{3,4} = \pm i\sqrt{\lambda^2 - \alpha_m^2}$$

$$Y(y) = C_1 \sin\delta_1 y + C_2 \cos\delta_1 y + C_3 \sinh\delta_2 y + C_4 \cosh\delta_2 y$$

$$\delta_1 = \sqrt{\lambda^2 - \alpha_m^2}, \qquad \delta_2 = \sqrt{\lambda^2 + \alpha_m^2}$$

Vibrations of Rectangular Plates

y = 0 and y = b are simply supported:

$$W(x,0)=0 \qquad Y(0)=0$$

$$W(x,b)=0 \qquad Y(b)=0$$

$$M_y(x,0) = -D\left(\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2}\right)\bigg|_{(x,0)} = 0 \quad \Longrightarrow \quad \frac{d^2Y(0)}{dy^2} = 0$$

$$M_y(x,b) = -D\left(\frac{\partial^2 W}{\partial y^2} + \nu\frac{\partial^2 W}{\partial x^2}\right)\bigg|_{(x,b)} = 0 \qquad \frac{d^2Y(b)}{dy^2} = 0$$

$$C_2 + C_4 = 0$$

$$C_1 \sin\delta_1 b + C_2 \cos\delta_1 b + C_3 \sinh\delta_2 b + C_4 \cosh\delta_2 b = 0$$

$$-\delta_1^2 C_2 + \delta_2^2 C_4 = 0$$

$$-C_1\delta_1^2 \sin\delta_1 b - C_2\delta_1^2 \cos\delta_1 b + C_3\delta_2^2 \sinh\delta_2 b + C_4\delta_2^2 \cosh\delta_2 b = 0$$

$$\Longrightarrow \quad C_4 = 0 \qquad C_2 = 0 \qquad C_3 = 0$$

Vibrations of Rectangular Plates

y = 0 and y = b are simply supported:

$$\sin\delta_1 b = 0 \qquad \delta_1 = \frac{n\pi}{b}, \qquad n = 1,\ 2,\ \ldots.$$

$$Y_n(y) = C_1 \sin\delta_1 y = C_1 \sin\frac{n\pi y}{b}$$

$$\delta_1 = \sqrt{\lambda^2 - \alpha_m^2}, \Longrightarrow \lambda_{mn}^2 = \alpha_m^2 + \beta_n^2, \qquad \beta_n = \frac{n\pi}{b}$$

$$\omega_{mn} = \lambda_{mn}^2\sqrt{\frac{D}{\rho h}} = (\alpha_m^2 + \beta_n^2)\sqrt{\frac{D}{\rho h}} = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]\sqrt{\frac{D}{\rho h}}, \qquad m, n = 1,\ 2, \ldots$$

$$W_{mn}(x, y) = C_{mn}\sin\alpha_m x\ \sin\beta_n y, \qquad m, n = 1,\ 2, \ldots$$

Vibrations of Rectangular Plates

y = 0 and y = b are clamped:

$$Y(0) = 0$$

$$\frac{dY}{dy}(0) = 0$$

$$Y(b) = 0$$

$$\frac{dY}{dy}(b) = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \delta_1 & 0 & \delta_2 & 0 \\ \sin \delta_1 b & \cos \delta_1 b & \sinh \delta_2 b & \cosh \delta_2 b \\ \delta_1 \cos \delta_1 b & -\delta_1 \sin \delta_1 b & \delta_2 \cosh \delta_2 b & \delta_2 \sinh \delta_2 b \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$2\delta_1\delta_2(\cos \delta_1 b \ \cosh \delta_2 b - 1) - \alpha_m^2 \sin \delta_1 b \ \sinh \delta_2 b = 0$$

$$Y_n(y) = C_n[(\cosh \delta_2 b - \cos \delta_1 b)\ (\delta_1 \sinh \delta_2 y - \delta_2 \sin \delta_1 y) - (\delta_1 \sinh \delta_2 b - \delta_2 \sin \delta_1 b)\ (\cosh \delta_2 y - \cos \delta_1 y)]$$

$$W_{mn}(x, y) = C_{mn} Y_n(y) \ \sin \alpha_m x$$

Vibrations of Rectangular Plates

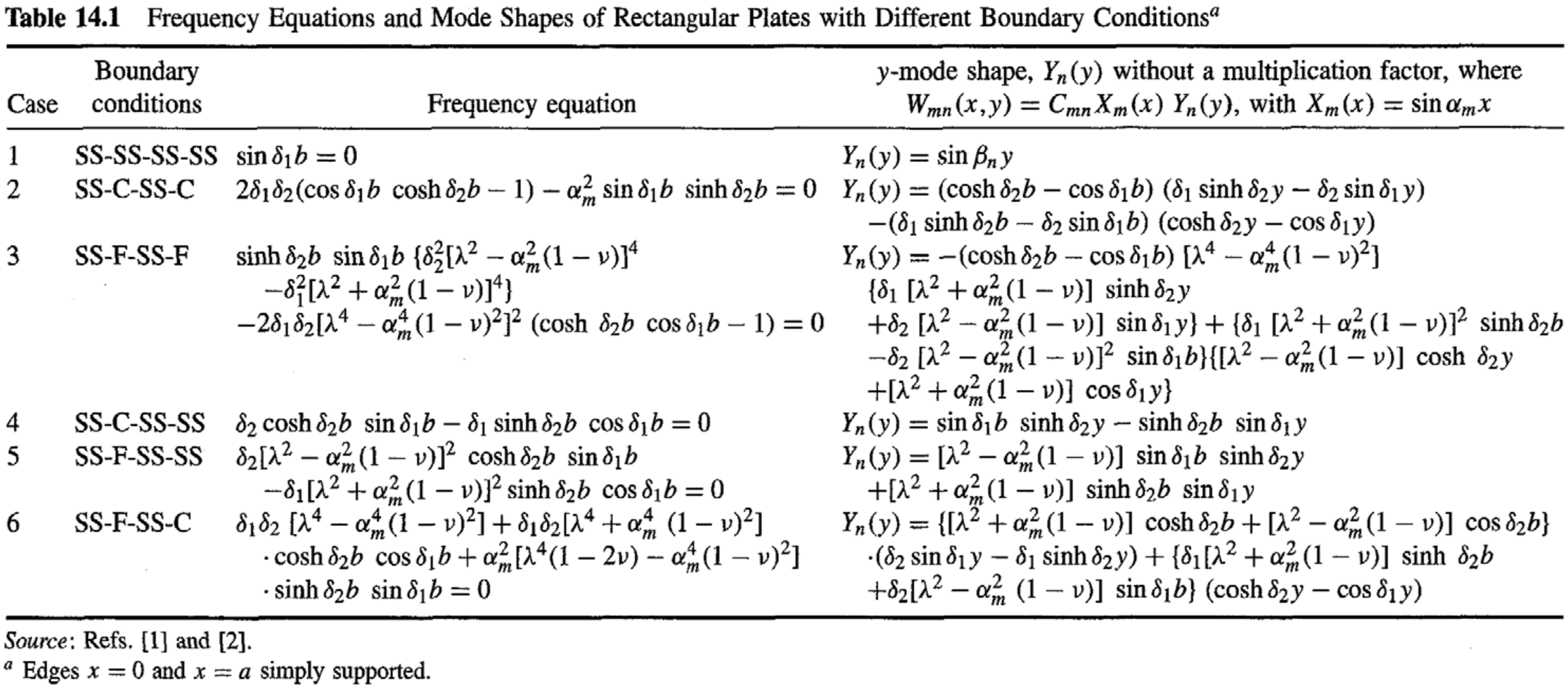


Table 14.1 Frequency Equations and Mode Shapes of Rectangular Plates with Different Boundary Conditions[a]

Case	Boundary conditions	Frequency equation	y-mode shape, $Y_n(y)$ without a multiplication factor, where $W_{mn}(x,y) = C_{mn} X_m(x)\, Y_n(y)$, with $X_m(x) = \sin \alpha_m x$
1	SS-SS-SS-SS	$\sin \delta_1 b = 0$	$Y_n(y) = \sin \beta_n y$
2	SS-C-SS-C	$2\delta_1\delta_2(\cos \delta_1 b \ \cosh \delta_2 b - 1) - \alpha_m^2 \sin \delta_1 b \ \sinh \delta_2 b = 0$	$Y_n(y) = (\cosh \delta_2 b - \cos \delta_1 b)\ (\delta_1 \sinh \delta_2 y - \delta_2 \sin \delta_1 y)$ $-(\delta_1 \sinh \delta_2 b - \delta_2 \sin \delta_1 b)\ (\cosh \delta_2 y - \cos \delta_1 y)$
3	SS-F-SS-F	$\sinh \delta_2 b \ \sin \delta_1 b \ \{\delta_2^2[\lambda^2 - \alpha_m^2(1-\nu)]^4$ $-\delta_1^2[\lambda^2 + \alpha_m^2(1-\nu)]^4\}$ $-2\delta_1\delta_2[\lambda^4 - \alpha_m^4(1-\nu)^2]^2\ (\cosh \ \delta_2 b \ \cos \delta_1 b - 1) = 0$	$Y_n(y) = -(\cosh \delta_2 b - \cos \delta_1 b)\ [\lambda^4 - \alpha_m^4(1-\nu)^2]$ $\{\delta_1\ [\lambda^2 + \alpha_m^2(1-\nu)]\ \sinh \delta_2 y$ $+\delta_2\ [\lambda^2 - \alpha_m^2(1-\nu)]\ \sin \delta_1 y\} + \{\delta_1\ [\lambda^2 + \alpha_m^2(1-\nu)]^2\ \sinh \delta_2 b$ $-\delta_2\ [\lambda^2 - \alpha_m^2(1-\nu)]^2\ \sin \delta_1 b\}\{[\lambda^2 - \alpha_m^2(1-\nu)]\ \cosh\ \delta_2 y$ $+[\lambda^2 + \alpha_m^2(1-\nu)]\ \cos \delta_1 y\}$
4	SS-C-SS-SS	$\delta_2 \cosh \delta_2 b \ \sin \delta_1 b - \delta_1 \sinh \delta_2 b \ \cos \delta_1 b = 0$	$Y_n(y) = \sin \delta_1 b \ \sinh \delta_2 y - \sinh \delta_2 b \ \sin \delta_1 y$
5	SS-F-SS-SS	$\delta_2[\lambda^2 - \alpha_m^2(1-\nu)]^2\ \cosh \delta_2 b \ \sin \delta_1 b$ $-\delta_1[\lambda^2 + \alpha_m^2(1-\nu)]^2 \sinh \delta_2 b \ \cos \delta_1 b = 0$	$Y_n(y) = [\lambda^2 - \alpha_m^2(1-\nu)]\ \sin \delta_1 b \ \sinh \delta_2 y$ $+[\lambda^2 + \alpha_m^2(1-\nu)]\ \sinh \delta_2 b \ \sin \delta_1 y$
6	SS-F-SS-C	$\delta_1\delta_2\ [\lambda^4 - \alpha_m^4(1-\nu)^2] + \delta_1\delta_2[\lambda^4 + \alpha_m^4\ (1-\nu)^2]$ $\cdot \cosh \delta_2 b \ \cos \delta_1 b + \alpha_m^2[\lambda^4(1-2\nu) - \alpha_m^4(1-\nu)^2]$ $\cdot \sinh \delta_2 b \ \sin \delta_1 b = 0$	$Y_n(y) = \{[\lambda^2 + \alpha_m^2(1-\nu)]\ \cosh \delta_2 b + [\lambda^2 - \alpha_m^2(1-\nu)]\ \cos \delta_2 b\}$ $\cdot(\delta_2 \sin \delta_1 y - \delta_1 \sinh \delta_2 y) + \{\delta_1[\lambda^2 + \alpha_m^2(1-\nu)]\ \sinh\ \delta_2 b$ $+\delta_2[\lambda^2 - \alpha_m^2\ (1-\nu)]\ \sin \delta_1 b\}\ (\cosh \delta_2 y - \cos \delta_1 y)$

Source: Refs. [1] and [2].

[a] Edges $x = 0$ and $x = a$ simply supported.

Vibrations of Rectangular Plates

Exact characteristic equations for some of classical boundary conditions of vibrating moderately thick rectangular plates

Shahrokh Hosseini Hashemi and M. Arsanjani ,International Journal of Solids and Structures Volume 42, Issues 3-4, February 2005, Pages 819-853

Exact solution for linear buckling of rectangular Mindlin plates

Shahrokh Hosseini-Hashemi, Korosh Khorshidi, and Marco Amabili, Journal of Sound and Vibration Volume 315, Issues 1-2, 5 August 2008, Pages 318-342

FORCED VIBRATION OF RECTANGULAR PLATES

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}(x,y)\eta_{mn}(t)$$

the normal modes

$$\int_0^a \int_0^b \rho h W_{mn}^2 \, dx \, dy = 1$$

FORCED VIBRATION OF RECTANGULAR PLATES

Using a modal analysis procedure:

$$\ddot{\eta}_{mn}(t) + \omega_{mn}^2 \eta_{mn}(t) = N_{mn}(t), \qquad m, n = 1, 2, \ldots$$

$$N_{mn}(t) = \int_0^a \int_0^b W_{mn}(x, y) f(x, y, t)\, dx\, dy$$

$$\eta_{mn}(t) = \eta_{mn}(0) \cos \omega_{mn} t + \frac{\dot{\eta}_{mn}(0)}{\omega_{mn}} \sin \omega_{mn} t$$

$$+ \frac{1}{\omega_{mn}} \int_0^t N_{mn}(\tau) \sin \omega_{mn}(t - \tau)\, d\tau$$

FORCED VIBRATION OF RECTANGULAR PLATES

The response of simply supported rectangular plates:

$$W_{mn}(x,y) = A_{1mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \qquad m, n = 1, 2, \ldots$$

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \eta_{mn}(0) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \left[\pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) t \right]$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\dot{\eta}_{mn}(0)(\rho h)^{1/2}}{\pi^2 (D)^{1/2}} \frac{1}{m^2/a^2 + n^2/b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

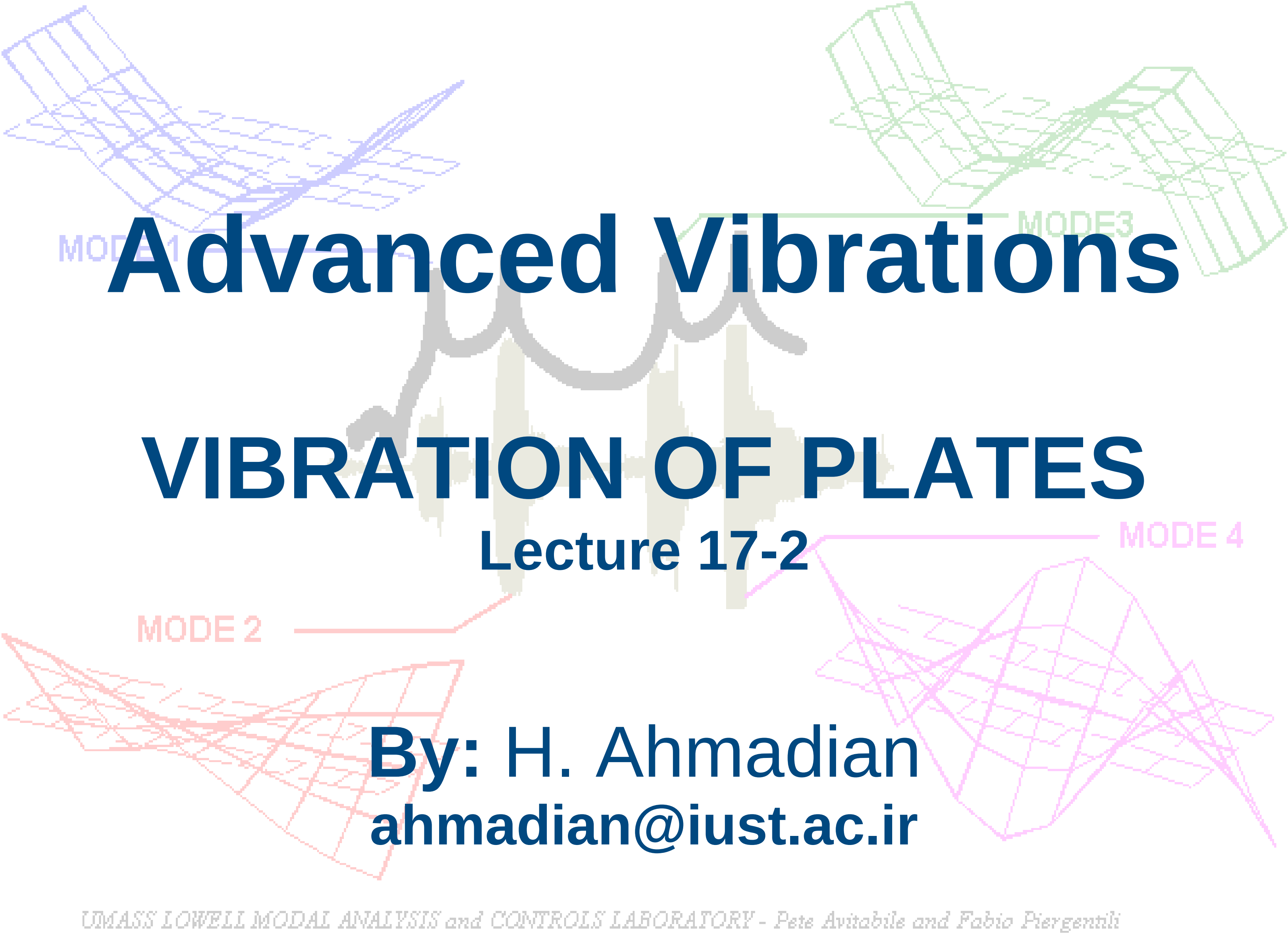
$$\sin \left[\pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) t \right]$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\rho h)^{1/2}}{\pi^2 D^{1/2}} \frac{1}{m^2/a^2 + n^2/b^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \int_0^t N_{mn}(\tau)$$

$$\sin \left[\pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) (t - \tau) \right] d\tau$$

$$A_{1mn} = 2/\sqrt{\rho h a b}$$

$$\omega_{mn} = \pi^2 \left(\frac{D}{\rho h} \right)^{1/2} \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]$$



UMASS LOWELL MODAL ANALYSIS and CONTROLS LABORATORY - Pete Avitabile and Fabio Piergentili

Advanced Vibrations

VIBRATION OF PLATES

Lecture 17-2

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EQUATION OF MOTION: Variational Approach

To develop the strain energy one may assume the state of stress in a thin plate as plane stress:

$$\pi_0 = \frac{1}{2}(\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{xy}\varepsilon_{xy})$$

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2}$$

$$\varepsilon_{xy} = -2z\frac{\partial^2 w}{\partial x\,\partial y}$$

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{xx} + \nu\varepsilon_{yy}) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}(\varepsilon_{yy} + \nu\varepsilon_{xx}) = -\frac{Ez}{1-\nu^2}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)$$

$$\sigma_{xy} = G\varepsilon_{xy} = \frac{E}{2(1+\nu)}\varepsilon_{xy} = -2Gz\frac{\partial^2 w}{\partial x\,\partial y} = -\frac{Ez}{1+\nu}\frac{\partial^2 w}{\partial x\,\partial y}$$

EQUATION OF MOTION: Variational Approach

$$\pi_0 = \frac{Ez^2}{2(1-\nu^2)}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]$$

$$\pi = \iiint\limits_V \pi_0\,dV$$

$$= \frac{D}{2}\iint\limits_A \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy$$

$$T = \frac{\rho h}{2}\iint\limits_A \left(\frac{\partial w}{\partial t}\right)^2 dx\,dy \qquad W = \iint\limits_A f\,w\,dx\,dy$$

EQUATION OF MOTION: Variational Approach

Extended Hamilton's principle can be written as:

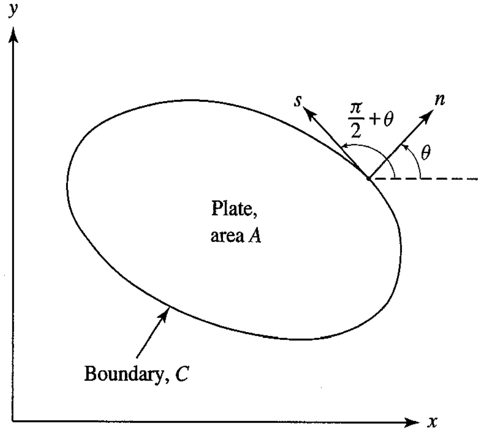
$$\delta \int_{t_1}^{t_2} \left(\frac{D}{2} \iint\limits_{A} \left\{ (\nabla^2 w)^2 - 2(1-\nu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx\,dy \right.$$

$$\left. - \frac{\rho h}{2} \iint\limits_{A} \left(\frac{\partial w}{\partial t} \right)^2 dx\,dy - \iint\limits_{A} f w\,dx\,dy \right) dt = 0$$

EQUATION OF MOTION: Variational Approach

$$I_1 = \delta \int_{t_1}^{t_2} \frac{D}{2} \iint_A (\nabla^2 w)^2 \, dx \, dy \, dt = D \int_{t_1}^{t_2} \iint_A \nabla^2 w \nabla^2 \delta w \, dx \, dy \, dt$$

$$= D \int_{t_1}^{t_2} \left\{ \iint_A \nabla^4 w \delta w \, dx \, dy + \int_C \left[\nabla^2 w \frac{\partial (\delta w)}{\partial n} - \delta w \frac{\partial (\nabla^2 w)}{\partial n} \right] dC \right\} dt$$

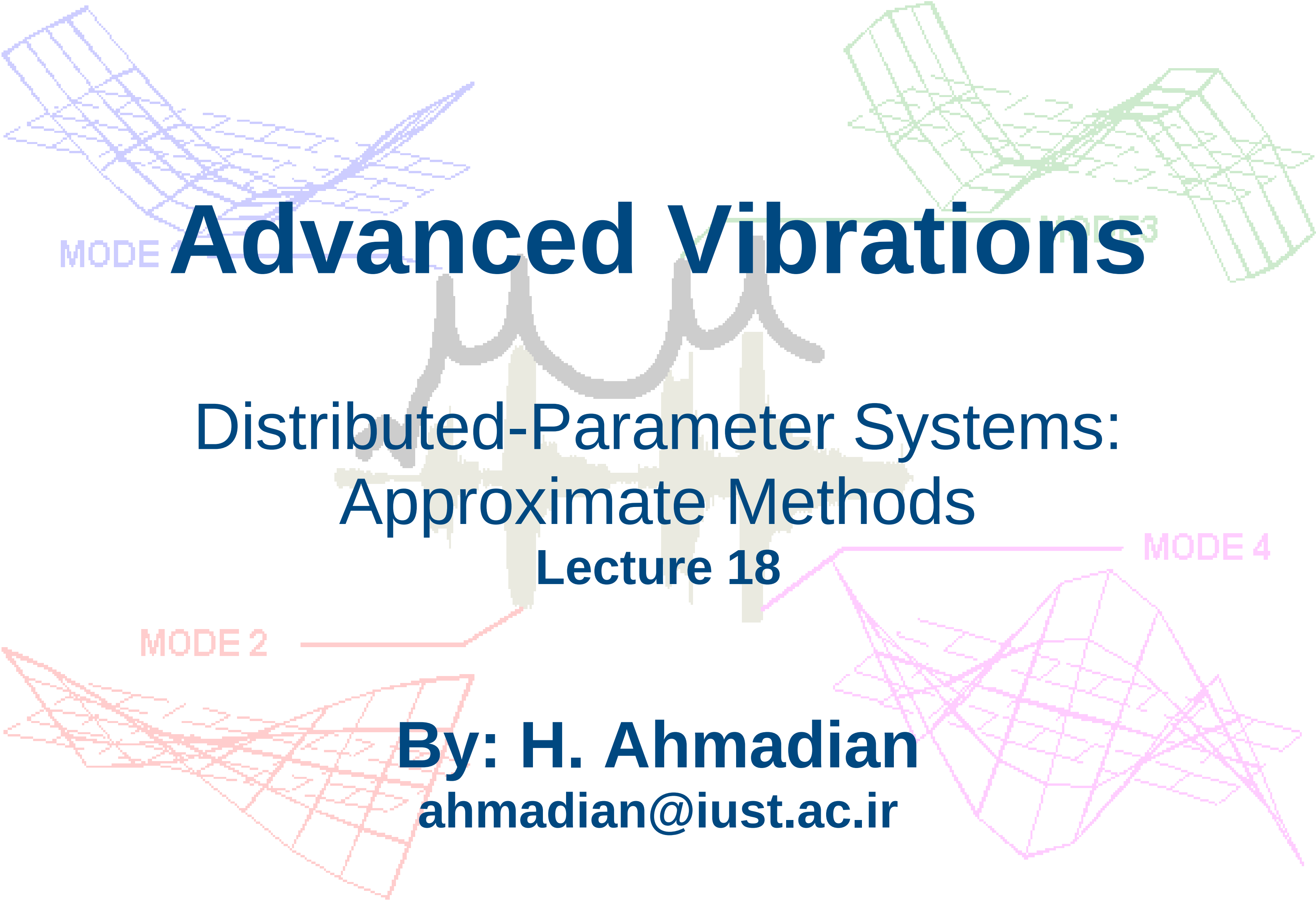


Green’s Theorem

$$\iint_A \left(\frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y} \right) dx \, dy = \oint_C \left(F_1 \, dx + F_2 \, dy \right)$$

EQUATION OF MOTION: Variational Approach

EQUATION OF MOTION: Variational Approach



Advanced Vibrations

Distributed-Parameter Systems: Approximate Methods

Lecture 18

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Distributed-Parameter Systems: Approximate Methods

- Rayleigh's Principle
- The Rayleigh-Ritz Method
- An Enhanced Rayleigh-Ritz Method
- The Assumed-Modes Method: System Response
- The Galerkin Method
- The Collocation Method

RAYLEIGH'S PRINCIPLE

The lowest eigenvalue is the minimum value that Rayleigh's quotient can take by letting the trial function *Y(x)* vary at will.

$$\lambda_1 = \omega_1^2 = \min R(Y) = R(Y_1)$$

The minimum value is achieved when Y(x) coincides with the lowest eigenfunction $Y_1(x)$.

RAYLEIGH'S PRINCIPLE

Consider the differential eigenvalue problem for a string in transverse vibration fixed at *x*=0 and supported by a spring of stiffness k at *x*=L.

$$-\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]=\lambda\rho(x)Y(x),\ 0<x<L,\ \lambda=\omega^2$$

$$Y(x)=0 \text{ at } x=0,\ T(x)\frac{dY(x)}{dx}+kY(x)=0 \text{ at } x=L$$

- Exact solutions are possible only in relatively few cases,
  - Most of them characterized by constant tension and uniform mass density.
- In seeking an approximate solution, sacrifices must be made, in the sense that something must be violated.
  - Almost always, one forgoes the exact solution of the differential equation, which will be satisfied only approximately,
  - But insists on satisfying both boundary conditions exactly.

RAYLEIGH'S PRINCIPLE

Rayleigh's principle, suggests a way of approximating the lowest eigenvalue, without solving the differential eigenvalue problem directly.

$$R(Y)=\lambda=\omega^2=\frac{-\int_0^L Y(x)\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]dx}{\int_0^L \rho(x)Y^2(x)dx}$$

Minimizing Rayleigh's quotient is equivalent to solving the differential equation in a weighted average sense, where the weighting function is *Y(x).*

RAYLEIGH'S PRINCIPLE

Boundary conditions do not appear explicitly in the weighted average form of Rayleigh's quotient.

To taken into account the characteristics of the system as much as possible, *the trial functions used in conjunction with the weighted average form of Rayleigh's quotient must satisfy all the boundary conditions of the problem.*

Comparison functions: trial functions that are as many times differentiable as the order of the system and satisfy all the boundary conditions.

RAYLEIGH'S PRINCIPLE

The trial functions must be from the class of comparison functions.

- The differentiability of the trial functions is seldom an issue.
- But the satisfaction of all the boundary conditions, particularly the satisfaction of the natural boundary conditions can be.

In view of this, we wish to examine the implications of violating the natural boundary conditions.

RAYLEIGH'S PRINCIPLE

$$-\int_0^L Y(x)\frac{d}{dx}\left[T(x)\frac{dY(x)}{dx}\right]dx = -Y(x)T(x)\frac{dY(x)}{dx}\bigg|_0^L + \int_0^L T(x)\left[\frac{dY(x)}{dx}\right]^2 dx$$

$$= \int_0^L T(x)\left[\frac{dY(x)}{dx}\right]^2 dx + kY^2(L)$$

$$R(Y) = \lambda = \omega^2 = \frac{V_{\max}}{T_{\text{ref}}}$$

$$V_{\max} = \frac{1}{2}\int_0^L T(x)\left[\frac{dY(x)}{dx}\right]^2 dx + \frac{1}{2}kY^2(L)$$

$$T_{\text{ref}} = \frac{1}{2}\int_0^L \rho(x)Y^2(x)dx$$

Rayligh’s *quotient* involves V_{max} and T_{ref}, which are defined for trial functions that are half as many times differentiable as the order of the system and

- need satisfy only the geometric boundary conditions,
- as the natural boundary conditions are accounted for in some fashion.

RAYLEIGH'S PRINCIPLE

- Trial functions that are half as many times differentiable as the order of the system and satisfy the geometric boundary conditions alone as *admissible functions.*
  - In using admissible functions in conjunction with the energy form of Rayleigh's quotient, the natural boundary conditions are still violated.
  - But, the deleterious effect of this violation is somewhat mitigated by the fact that the energy form of Rayleigh's quotient, includes contributions to V_{max} from springs at boundaries and to T_{ref} from masses at boundaries.
- But if comparison functions are available, then their use is preferable over the use of admissible functions, because the results are likely to be more accurate.

Example: Lowest natural frequency of the fixed-free tapered rod in axial vibration

$$m(x)=\frac{6m}{5}\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right],\quad EA(x)=\frac{6EA}{5}\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

The 1st mode of a uniform clamped-free rod as a trial function: $U(x)=\sin\frac{\pi x}{2L}$

A comparison function

$$R(U)=\omega^2=\frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L m(x)U^2(x)dx}=\frac{EA}{m}\left(\frac{\pi}{2L}\right)^2\frac{(L/12\pi^2)(5\pi^2+6)}{(L/12\pi^2)(5\pi^2-6)}$$

$$\omega=1.7749\sqrt{\frac{EA}{mL^2}}$$

THE RAYLEIGH-RITZ METHOD

The method was developed by Ritz as an extension of Rayleigh's energy method.

- Although Rayleigh claimed that the method originated with him, the form in which the method is generally used is due to Ritz.

The first step in the Rayleigh-Ritz method is to construct the *minimizing sequence*:

$$Y^{(1)}(x) = a_1\phi_1(x)$$

undetermined coefficients

$$Y^{(2)}(x) = a_1\phi_1(x) + a_2\phi_2(x) = \sum_{i=1}^{2} a_i\phi_i(x)$$

independent trial functions

$$\vdots$$

$$Y^{(n)}(x) = a_1\phi_1(x) + a_2\phi_2(x) + \ldots + a_n\phi_n(x) = \sum_{i=1}^{n} a_i\phi_i(x)$$

THE RAYLEIGH-RITZ METHOD

$$\lambda^{(n)} = R(Y^{(n)}) = R(a_1, a_2, \ldots, a_n)$$

$$\delta R = \frac{\partial R}{\partial a_1}\delta a_1 + \frac{\partial R}{\partial a_2}\delta a_2 + \ldots + \frac{\partial R}{\partial a_n}\delta a_n = 0$$

The independence of the trial functions implies the independence of the coefficients, which in turn implies the independence of the variations

$$\delta a_1, \delta a_2, \ldots, \delta a_n \Longrightarrow$$

$$\frac{\partial R}{\partial a_i} = 0, \; i = 1, 2, \ldots, n$$

THE RAYLEIGH-RITZ METHOD

$$\lambda^{(n)} = R(a_1, a_2, \ldots, a_n) = \frac{N(a_1, a_2, \ldots, a_n)}{D(a_1, a_2, \ldots, a_n)}$$

$$\frac{\partial R}{\partial a_i} = \frac{(\partial N/\partial a_i)D - (\partial D/\partial a_i)N}{D^2} = \frac{1}{D}\left(\frac{\partial N}{\partial a_i} - \frac{N}{D}\frac{\partial D}{\partial a_i}\right)$$

$$= \frac{1}{D}\left(\frac{\partial N}{\partial a_i} - \lambda^{(n)}\frac{\partial D}{\partial a_i}\right) = 0, \; i = 1, 2, \ldots, n$$

$$\frac{\partial N}{\partial a_i} - \lambda^{(n)}\frac{\partial D}{\partial a_i} = 0, \; i = 1, 2, \ldots, n$$

Solving the equations amounts to determining the coefficients, as well as to determining $\lambda^{(n)}$

THE RAYLEIGH-RITZ METHOD

To illustrate the Rayleigh-Ritz process, we consider the differential eigenvalue problem for the string in transverse vibration:

$$\begin{aligned} N = V_{\max} &= \frac{1}{2}\int_0^L T(x)\left[\frac{dY^{(n)}(x)}{dx}\right]^2 dx + \frac{1}{2}k[Y^{(n)}(L)]^2 \\ &= \frac{1}{2}\int_0^L T(x)\sum_{i=1}^n a_i \frac{d\phi_i(x)}{dx}\sum_{j=1}^n a_j \frac{d\phi_j(x)}{dx} dx + \frac{1}{2}k\sum_{i=1}^n a_i\phi_i(L)\sum_{j=1}^n a_j\phi_j(L) \\ &= \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n a_i a_j\left[\int_0^L T(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx + k\phi_i(L)\phi_j(L)\right] \\ &= \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n k_{ij}a_i a_j \end{aligned}$$

$$D = T_{\text{ref}} = \frac{1}{2}\int_0^L \rho(x)[Y^{(n)}(x)]^2 dx = \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n m_{ij}a_i a_j$$

THE RAYLEIGH-RITZ METHOD

$$N = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} k_{rs} a_r a_s \qquad D = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} m_{rs} a_r a_s$$

$$\frac{\partial N}{\partial a_i} = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} k_{rs}\left(\frac{\partial a_r}{\partial a_i} a_s + a_r \frac{\partial a_s}{\partial a_i}\right) = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} k_{rs}(\delta_{ri} a_s + a_r \delta_{si})$$

$$= \frac{1}{2}\left(\sum_{s=1}^{n} k_{is} a_s + \sum_{r=1}^{n} k_{ri} a_r\right) = \sum_{s=1}^{n} k_{is} a_s$$

$$\frac{\partial D}{\partial a_i} = \sum_{s=1}^{n} m_{is} a_s$$

$$\sum_{s=1}^{n} k_{is} a_s = \lambda^{(n)} \sum_{s=1}^{n} m_{is} a_s, \quad i = 1, 2, \ldots, n$$

$$K^{(n)}\mathbf{a}^{(n)} = \lambda^{(n)} M^{(n)}\mathbf{a}^{(n)}$$

Example : Solve the eigenvalue problem for the fixed-free tapered rod in axial vibration

The comparison functions $\phi_i(x) = \sin\frac{(2i-1)\pi x}{2L}, \ i = 1, 2, \ldots, n$

$$V_{\max} = \frac{1}{2}\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx \qquad T_{\text{ref}} = \frac{1}{2}\int_0^L m(x)U^2(x)dx$$

$$U^{(n)}(x) = \sum_{i=1}^{n} a_i^{(n)}\phi_i(x) = \sum_{i=1}^{n} a_i^{(n)}\sin\frac{(2i-1)\pi x}{2L}$$

$$V_{\max} \cong \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}^{(n)} a_i^{(n)} a_j^{(n)} \qquad T_{\text{ref}} \cong \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}^{(n)} a_i^{(n)} a_j^{(n)}$$

Example :

$$k_{ij}^{(n)} = \int_0^L EA(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} dx$$

$$= \frac{6EA}{5} \frac{(2i-1)\pi}{2L} \frac{(2j-1)\pi}{2L} \int_0^L \left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] \cos\frac{(2i-1)\pi x}{2L} \cos\frac{(2j-1)\pi x}{2L} dx,$$

$$m_{ij}^{(n)} = \int_0^L m(x)\phi_i(x)\phi_j(x)dx$$

$$= \frac{6m}{5} \int_0^L \left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] \sin\frac{(2i-1)\pi x}{2L} \sin\frac{(2j-1)\pi x}{2L} dx, \quad i, j = 1, 2, \ldots, n$$

Example : *n* = 2

$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 1.383701 & 0.337500 \\ 0.337500 & 11.253305 \end{bmatrix} \qquad M^{(2)} = mL\begin{bmatrix} 0.439207 & 0.075991 \\ 0.075991 & 0.493245 \end{bmatrix}$$

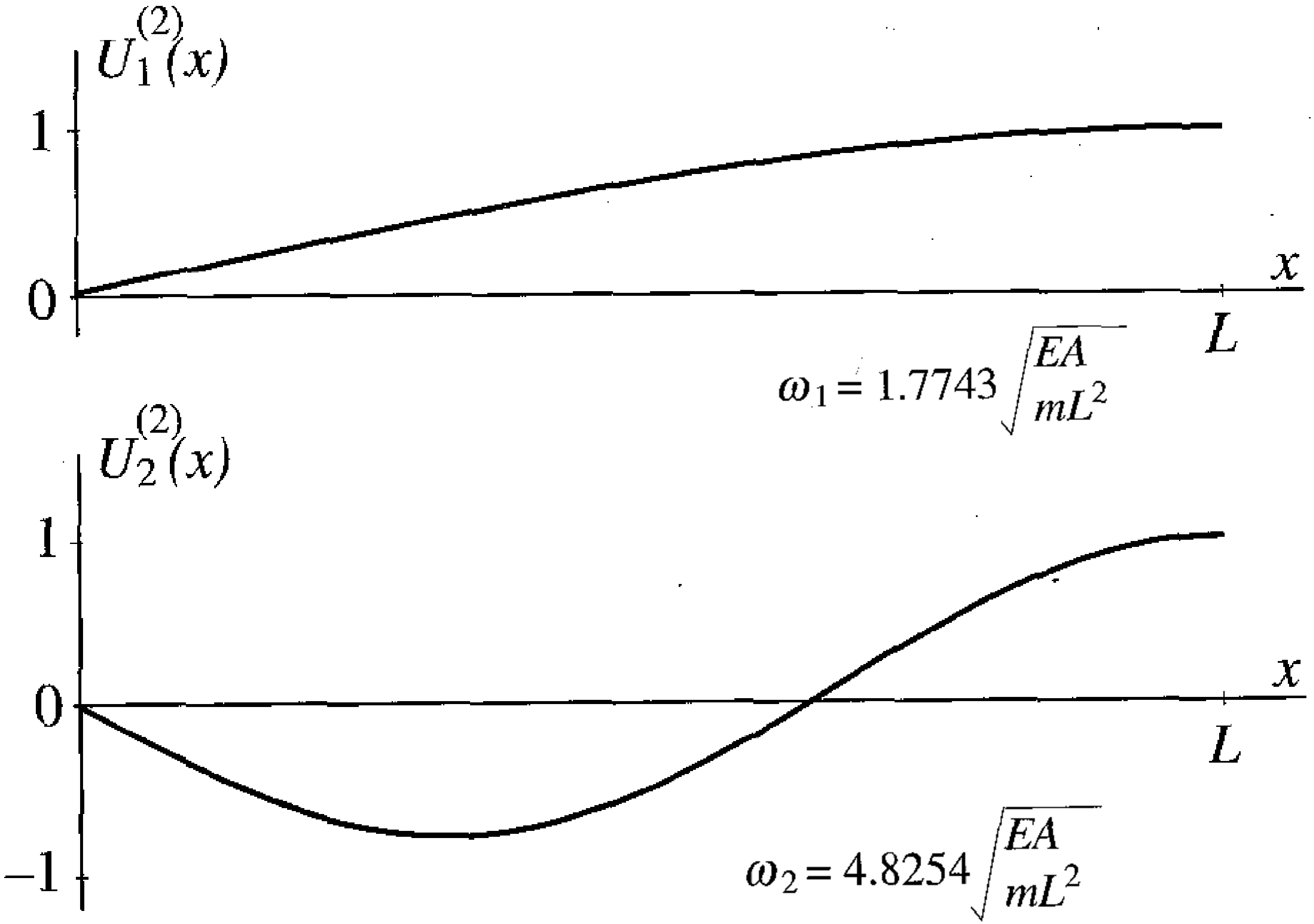
$$\omega_1^{(2)} = 1.774312\sqrt{\frac{EA}{mL^2}}, \ \mathbf{a}_1^{(2)} = (mL)^{-1/2}\begin{bmatrix} 1.511481 \\ -0.015311 \end{bmatrix}$$

$$\omega_2^{(2)} = 4.825444\sqrt{\frac{EA}{mL^2}}, \ \mathbf{a}_2^{(2)} = (mL)^{-1/2}\begin{bmatrix} -0.233683 \\ 1.443148 \end{bmatrix}$$

$$U_1^{(2)}(x) = 1.511481\sin\frac{\pi x}{2L} - 0.015311\sin\frac{3\pi x}{2L}$$

$$U_2^{(2)}(x) = -0.233683\sin\frac{\pi x}{2L} + 1.443148\sin\frac{3\pi x}{2L}$$

Example : $n = 2$



$$\omega_1 = 1.7743\sqrt{\frac{EA}{mL^2}}$$

$$\omega_2 = 4.8254\sqrt{\frac{EA}{mL^2}}$$

Example : *n = 3*

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} 1.383701 & 0.337500 & -0.104167 \\ 0.337500 & 11.253305 & 2.109375 \\ -0.104167 & 2.109375 & 30.992514 \end{bmatrix}$$

$$M^{(3)} = mL\begin{bmatrix} 0.439207 & 0.075991 & -0.021953 \\ 0.075991 & 0.493245 & 0.064592 \\ -0.021953 & 0.064592 & 0.497568 \end{bmatrix}$$

$$\omega_1^{(3)} = 1.774247\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_1^{(3)} = (mL)^{-1/2}\begin{bmatrix} 1.511715 \\ -0.015872 \\ 0.002829 \end{bmatrix}$$

$$\omega_2^{(3)} = 4.822187\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_2^{(3)} = (mL)^{-1/2}\begin{bmatrix} -0.236352 \\ 1.448321 \\ -0.040348 \end{bmatrix}$$

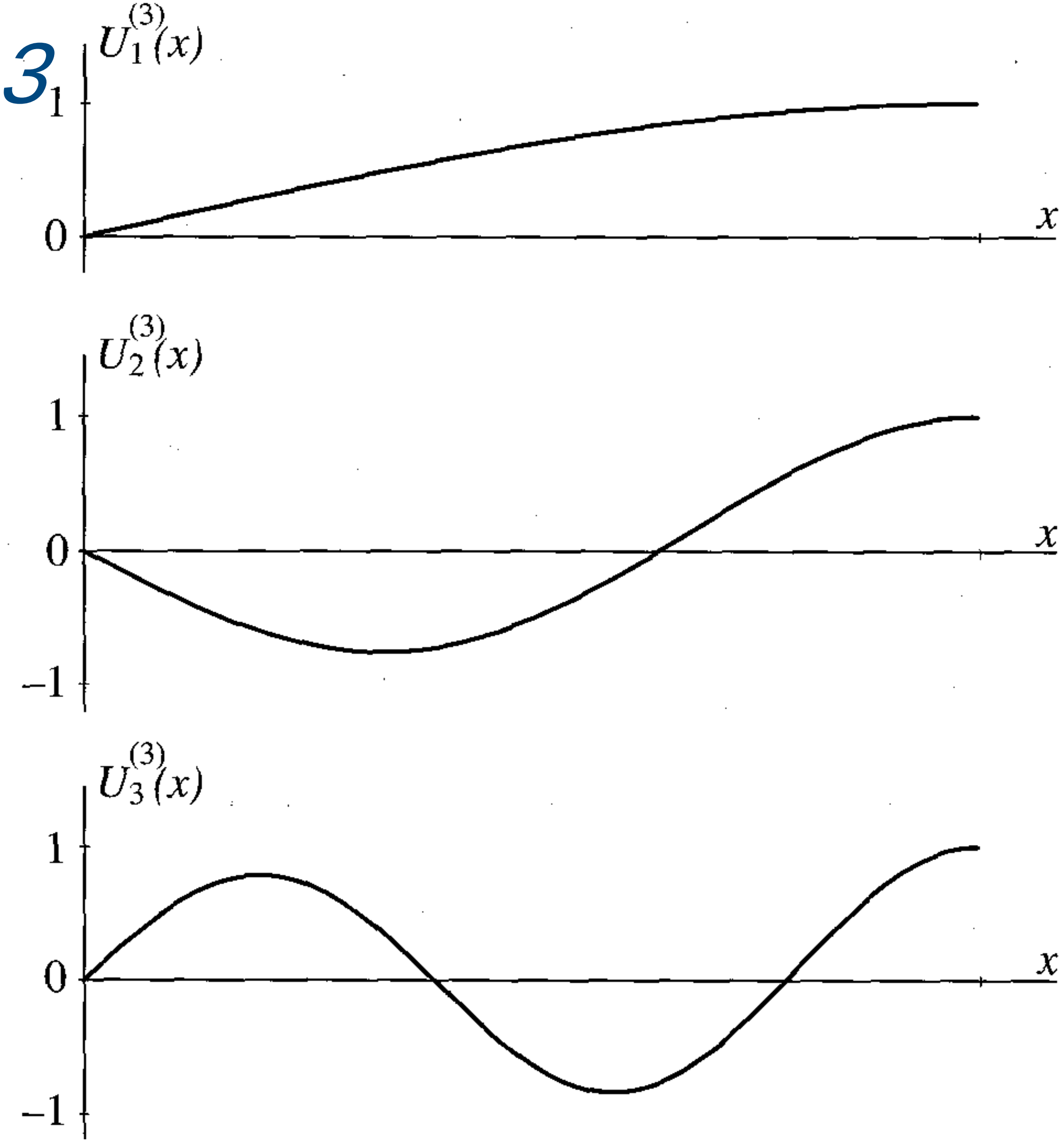
$$\omega_3^{(3)} = 7.931607\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_3^{(3)} = (mL)^{-1/2}\begin{bmatrix} 0.097373 \\ -0.163450 \\ 1.432793 \end{bmatrix}$$

Example : *n* = 3

$$\omega_1 = 1.7742\sqrt{\frac{EA}{mL^2}}$$

$$\omega_2 = 4.8222\sqrt{\frac{EA}{mL^2}}$$

$$\omega_3 = 7.9316\sqrt{\frac{EA}{mL^2}}$$



Example :

The Ritz eigenvalues for the two approximations are:

$$\lambda_1^{(2)} = 3.148183EA/mL^2,\ \lambda_2^{(2)} = 23.284913EA/mL^2$$

$$\lambda_1^{(3)} = 3.147951EA/mL^2,\ \lambda_2^{(3)} = 23.253490EA/mL^2,\ \lambda_3^{(3)} = 62.910394EA/mL^2$$

- The improvement in the first two Ritz natural frequencies is very small,
  - indicates the chosen comparison functions resemble very closely the actual natural modes.
- Convergence to the lowest eigenvalue with six decimal places accuracy is obtained with 11 terms: $\lambda_1^{(11)} = 3.147888EA/mL^2$

Truncation

Approximation of a system with an infinite number of DOFs by a discrete system with n degrees of freedom implies truncation:

$$a_{n+1} = a_{n+2} = \ldots = 0$$

Constraints tend to increase the stiffness of a system:

$$\lambda_r^{(n)} \geq \lambda_r\,,\ r = 1, 2, \ldots, n$$

The nature of the Ritz eigenvalues requires further elaboration.

Truncation

A question of particular interest is how the eigenvalues $\lambda_r^{(n+1)} \ (r = 1, 2, \ldots, n+1)$ of the (n +1)-DOF approximation relate to the eigenvalues $\lambda_r^{(n)} \ (r = 1, 2, \ldots, n)$ of the n-DOF approximation.

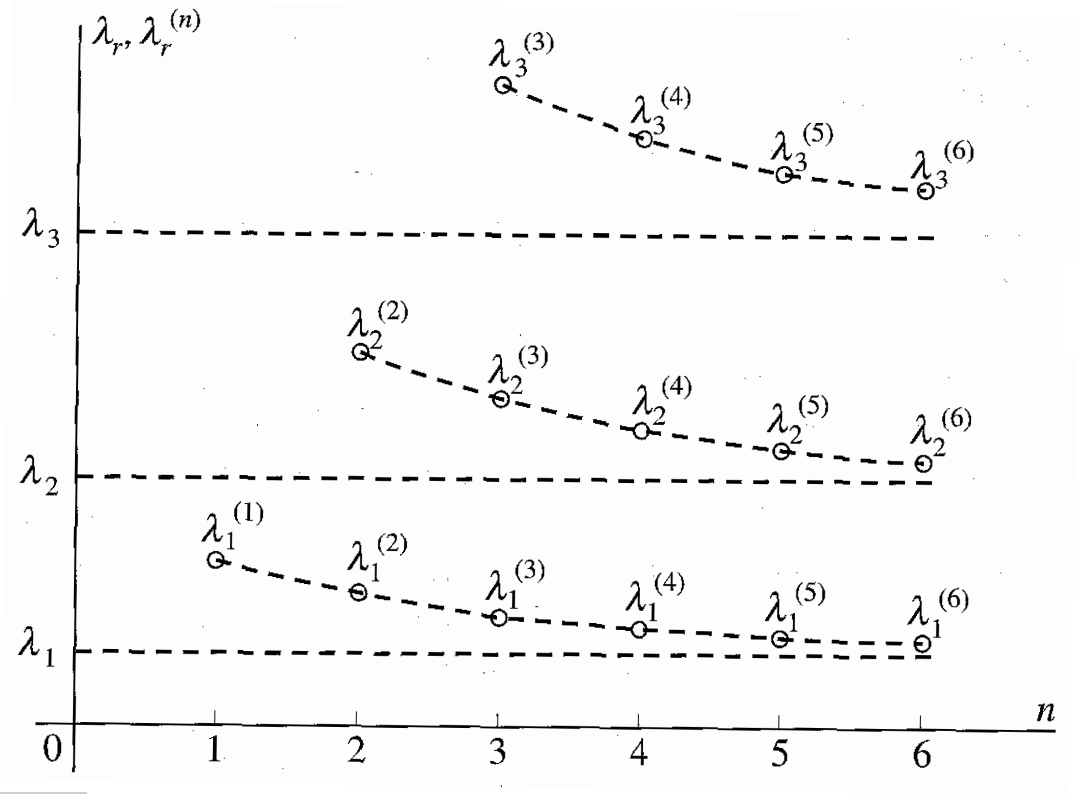
We observe that the extra term in series does not affect the mass and stiffness coefficients computed on the basis of an n-term series (embedding property):

$$M^{(n+1)} = \begin{bmatrix} \begin{matrix} & M^{(n)} & \\ & & \end{matrix} & \begin{matrix} x \\ x \end{matrix} \\ \begin{matrix} x & & x \end{matrix} & x \end{bmatrix}, \quad K^{(n+1)} = \begin{bmatrix} \begin{matrix} & K^{(n)} & \\ & & \end{matrix} & \begin{matrix} x \\ x \end{matrix} \\ \begin{matrix} x & & x \end{matrix} & x \end{bmatrix}$$

Truncation

For matrices with embedding property the eigenvalues satisfy the *separation theorem:*

$$\lambda_1^{(n+1)} \le \lambda_1^{(n)} \le \lambda_2^{(n+1)} \le \lambda_2^{(n)} \le \dots \le \lambda_n^{(n+1)} \le \lambda_n^{(n)} \le \lambda_{n+1}^{(n+1)}$$



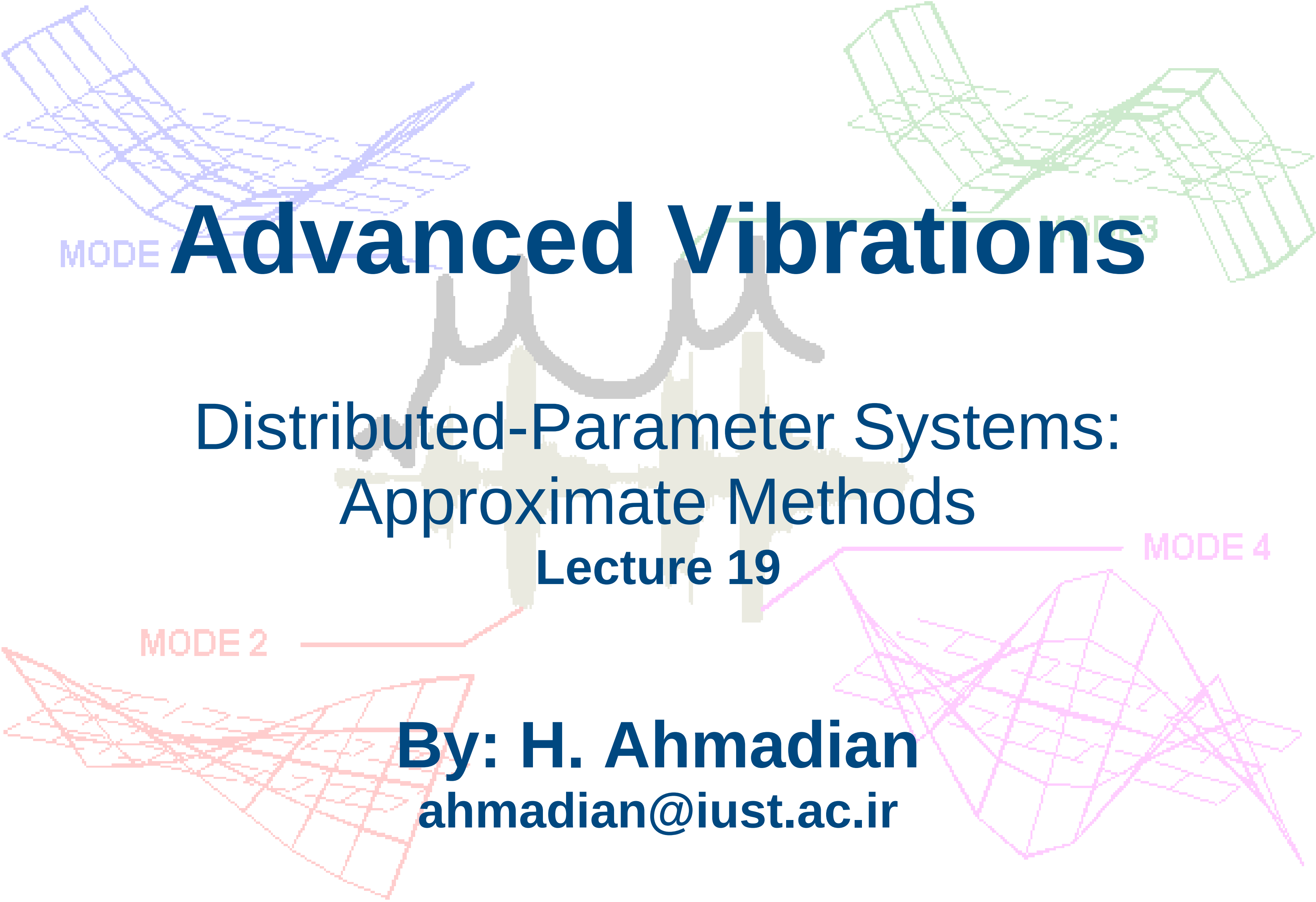
$$\lambda_r^{(n+1)} \le \lambda_r^{(n)}, \; r = 1, 2, \dots, n$$

$$\lim_{n \to \infty} \lambda_r^{(n)} = \lambda_r, \; r = 1, 2, \dots, n$$

Distributed-Parameter Systems: Approximate Methods

- Rayleigh's Principle
- The Rayleigh-Ritz Method
- An Enhanced Rayleigh-Ritz Method
- The Assumed-Modes Method: System Response
- The Galerkin Method
- The Collocation Method

Advanced Vibrations



Distributed-Parameter Systems: Approximate Methods

Lecture 19

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Distributed-Parameter Systems: Approximate Methods

- Rayleigh's Principle
- The Rayleigh-Ritz Method
- An Enhanced Rayleigh-Ritz Method
- The Assumed-Modes Method: System Response
- The Galerkin Method
- The Collocation Method

Rayleigh-Ritz method (contd.)

- How to choose suitable comparison functions, or admissible functions:
  - ✓ the requirement that all boundary conditions, or merely the geometric boundary conditions be satisfied is too broad to serve as a guideline.
- There may be several sets of functions that could be used and the rate of convergence tends to vary from set to set.
- It is imperative that the functions be from a complete set, because otherwise convergence may not be possible:
  - ✓ power series, trigonometric functions, Bessel functions, Legendre polynomials, etc.

Rayleigh-Ritz method

- Extreme care must be exercised when the end involves a discrete component, such as a spring or a lumped mass,
  - As an illustration, we consider a rod in axial vibration fixed at *x=0* and restrained by a spring of stiffness *k* at *x=L:*

$$EA(x)\frac{dU(x)}{dx}+kU(x)=0,\ x=L$$

- If we choose as admissible functions the eigenfunctions of a uniform fixed-free rod, then the rate of convergence will be very poor:

$$\phi_i(x)=\sin\frac{(2i-1)\pi x}{2L},\ i=1,2,\dots,n$$

- The rate of convergence can be vastly improved by using comparison functions:

$$\phi_i(x)=\sin\beta_i x,\ i=1,2,\dots,n$$

$$EA(L)\beta_i\cos\beta_i L+k\sin\beta_i L=0,$$

Rayleigh-Ritz method

Example : Consider the case in which the end x = L of the rod of previous example is restrained by a spring of stiffness k = *EA/L* and obtain the solution of the eigenvalue problem derived by the Rayleigh-Ritz method:

1) Using admissible functions $\phi_i(x) = \sin(2i-1)\pi x/2$
2) Using the comparison functions $\phi_i(x) = \sin \beta_i x,$

$$m(x) = \frac{6m}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right], \quad EA(x) = \frac{6EA}{5}\left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

Example: Using Admissible Functions

$$k_{ij}^{(n)} = \int_0^L EA(x)\frac{d\phi_i(x)}{dx}\frac{d\phi_j(x)}{dx}dx + k\phi_i(L)\phi_j(L)$$

$$= \frac{6EA}{5}\frac{(2i-1)\pi}{2L}\frac{(2j-1)\pi}{2L}\int_0^L\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\cos\frac{(2i-1)\pi x}{2L}\cos\frac{(2j-1)\pi x}{2L}dx$$

$$+\frac{EA}{L}\sin\frac{(2i-1)\pi}{2}\sin\frac{(2j-1)\pi}{2}, \qquad i,j=1,2,\ldots,n$$

$$m_{ij}^{(n)} = \int_0^L m(x)\phi_i(x)\phi_j(x)dx$$

$$= \frac{6m}{5}\int_0^L\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\sin\frac{(2i-1)\pi x}{2L}\sin\frac{(2j-1)\pi x}{2L}dx, \qquad i,j=1,2,\ldots,n$$

Example: Using Admissible Functions, Setting n=2

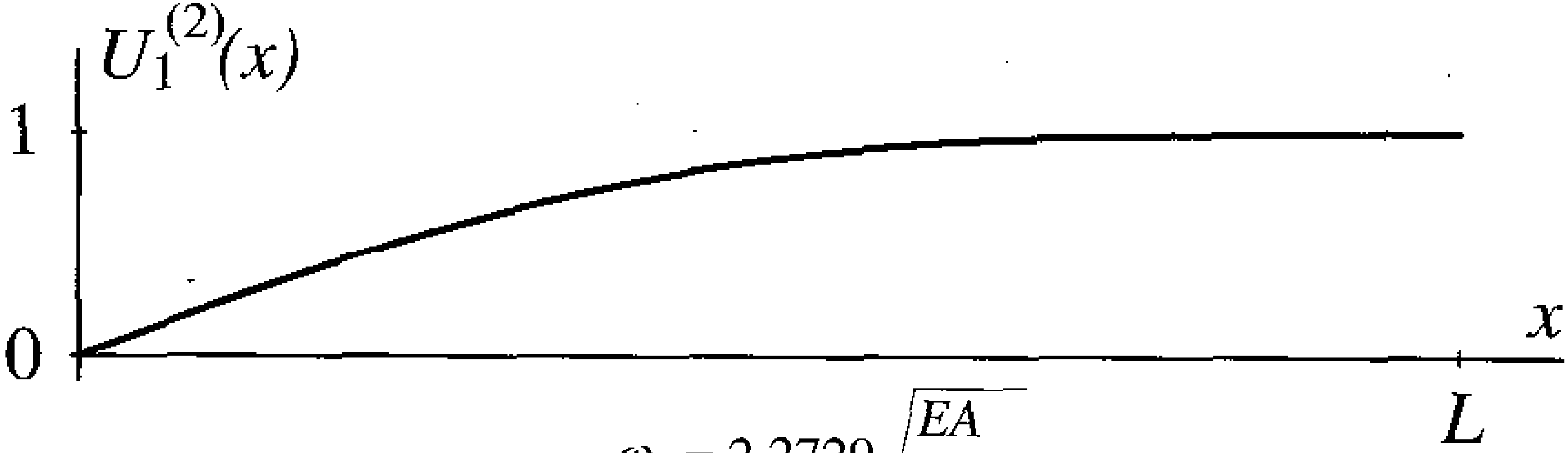
$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 2.383701 & -0.662500 \\ -0.662500 & 12.253305 \end{bmatrix} \quad M^{(2)} = mL\begin{bmatrix} 0.439207 & 0.075991 \\ 0.075991 & 0.493245 \end{bmatrix}$$

$$\omega_1^{(2)} = 2.272911\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_1^{(2)} = (mL)^{-1/2}\begin{bmatrix} 1.471927 \\ 0.160018 \end{bmatrix}$$

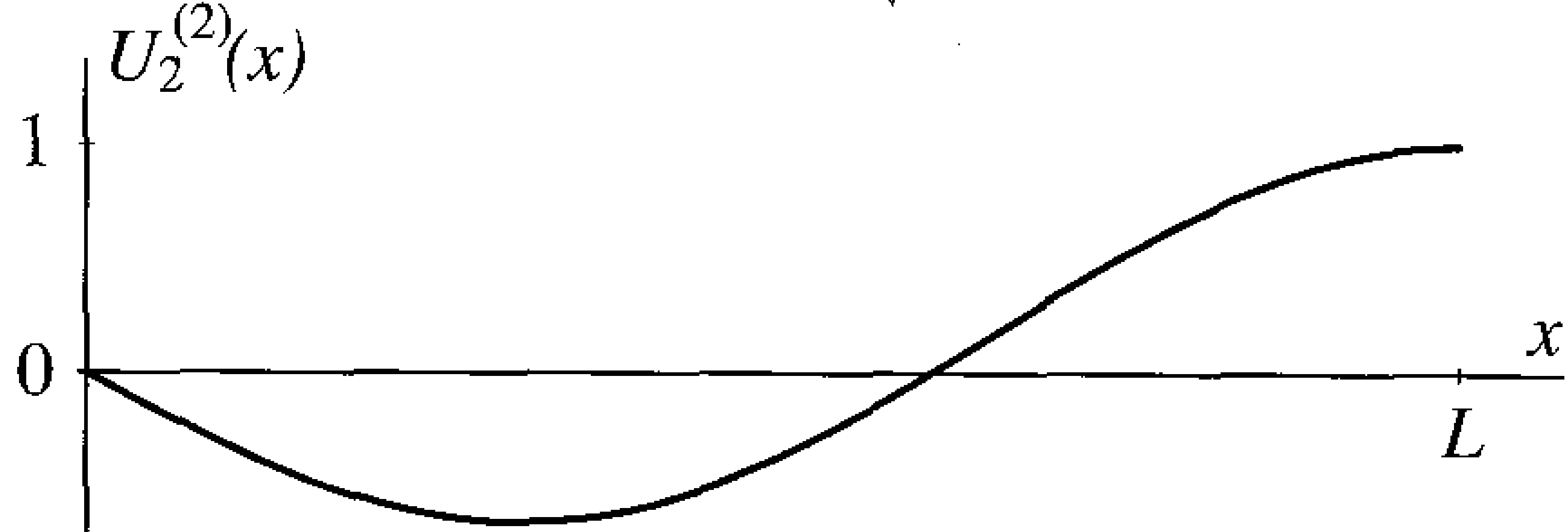
$$\omega_2^{(2)} = 5.139049\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_2^{(2)} = (mL)^{-1/2}\begin{bmatrix} -0.415467 \\ 1.434331 \end{bmatrix}$$

$$U_1^{(2)}(x) = 1.471927\sin\frac{\pi x}{2L} + 0.160018\sin\frac{3\pi x}{2L}$$

$$U_2^{(2)}(x) = -0.415467\sin\frac{\pi x}{2L} + 1.434331\sin\frac{3\pi x}{2L}$$



$$\omega_1 = 2.2729\sqrt{\frac{EA}{mL^2}}$$



$$\omega_2 = 5.1390\sqrt{\frac{EA}{mL^2}}$$

Example: Using Admissible Functions, Setting n=3

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} 2.383701 & -0.662500 & 0.895833 \\ -0.662500 & 12.253305 & 1.109375 \\ 0.895833 & 1.109375 & 31.992514 \end{bmatrix} \quad M^{(3)} = mL\begin{bmatrix} 0.439207 & 0.075991 & -0.021953 \\ 0.075991 & 0.493245 & 0.064592 \\ -0.021953 & 0.064592 & 0.497568 \end{bmatrix}$$

$$\omega_1^{(3)} = 2.253516\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_1^{(3)} = (mL)^{-1/2}\begin{bmatrix} 1.468344 \\ 0.162283 \\ -0.054500 \end{bmatrix}$$

$$\omega_2^{(3)} = 5.128225\sqrt{\frac{EA}{mL^2}}, \quad a_2^{(3)} = (mL)^{-1/2}\begin{bmatrix} -0.400771 \\ 1.422469 \\ 0.075563 \end{bmatrix}$$

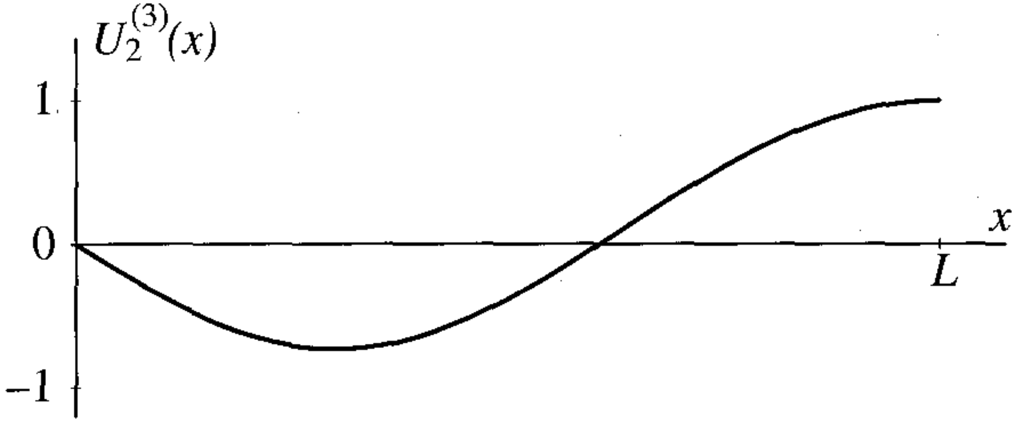
$$\omega_3^{(3)} = 8.131483\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_3^{(3)} = (mL)^{-1/2}\begin{bmatrix} 0.184319 \\ -0.273582 \\ 1.430333 \end{bmatrix}$$

$$U_1^{(3)} = 1.468344\sin\frac{\pi x}{2L} + 0.162283\sin\frac{3\pi x}{2L} - 0.054500\sin\frac{5\pi x}{2L}$$

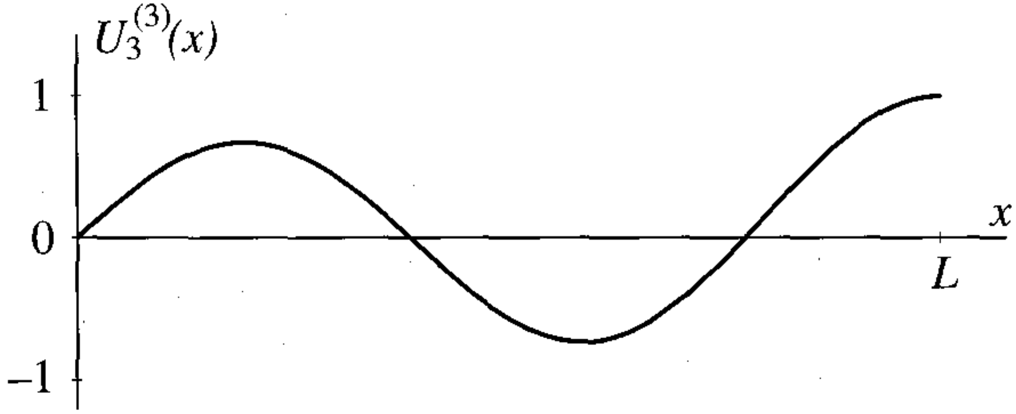
$$U_2^{(3)} = -0.400771\sin\frac{\pi x}{2L} + 1.422469\sin\frac{3\pi x}{2L} + 0.075563\sin\frac{5\pi x}{2L}$$

$$U_3^{(3)} = 0.184319\sin\frac{\pi x}{2L} - 0.273582\sin\frac{3\pi x}{2L} + 1.430333\sin\frac{5\pi x}{2L}$$

$$\omega_1 = 2.2535\sqrt{\frac{EA}{mL^2}}$$



$$\omega_2 = 5.1282\sqrt{\frac{EA}{mL^2}}$$



$$\omega_3 = 8.1315\sqrt{\frac{EA}{mL^2}}$$

Example: Using Admissible Functions,

- The convergence using admissible functions is extremely slow.
- Using n = 30, none of the natural frequencies has reached convergence with six decimal places accuracy:

$$\omega_1^{(30)} = 2.218950\sqrt{EA/mL^2},$$

$$\omega_2^{(30)} = 5.102324\sqrt{EA/mL^2},$$

$$\omega_3^{(30)} = 8.118398\sqrt{EA/mL^2}$$

Example: Using Comparison Function

$$\phi_i(x) = \sin \beta_i x, \quad i = 1, 2, \ldots, n$$

$$\beta_1 L = 2.215707, \ \beta_2 L = 5.032218, \ \beta_3 L = 8.057941, \ldots$$

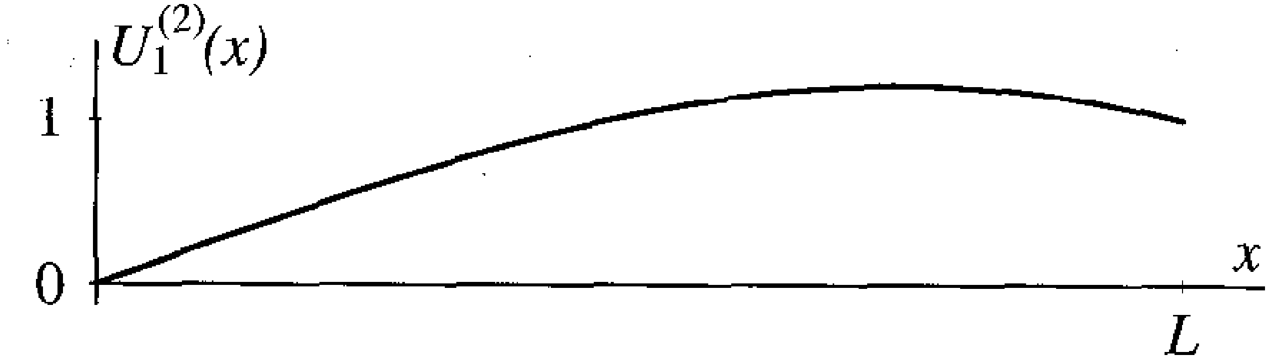
$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 2.783074 & 0.836697 \\ 0.836697 & 13.223631 \end{bmatrix} \quad M^{(2)} = mL\begin{bmatrix} 0.563196 & 0.085462 \\ 0.085462 & 0.513392 \end{bmatrix}$$

$$\omega_1^{(2)} = 2.216471\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_1^{(2)} = (mL)^{-1/2}\begin{bmatrix} 1.339519 \\ -0.052177 \end{bmatrix}$$

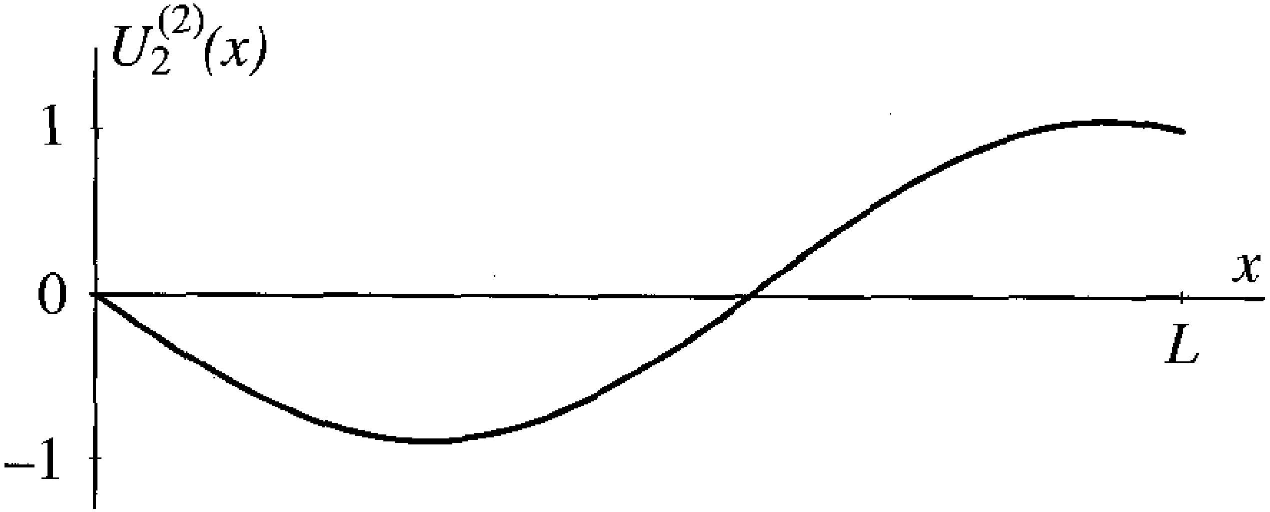
$$\omega_2^{(2)} = 5.106305\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_2^{(2)} = (mL)^{-1/2}\begin{bmatrix} -0.165180 \\ 1.412652 \end{bmatrix}$$

$$U_1^{(2)}(x) = 1.339519 \sin 2.215707\frac{x}{L} - 0.052177 \sin 5.032218\frac{x}{L}$$

$$U_2^{(2)}(x) = -0.165180 \sin 2.215707\frac{x}{L} + 1.412652 \sin 5.032218\frac{x}{L}$$



$$\omega_1 = 2.2165\sqrt{\frac{EA}{mL^2}}$$



$$\omega_2 = 5.1063\sqrt{\frac{EA}{mL^2}}$$

Example: Using Comparison Function

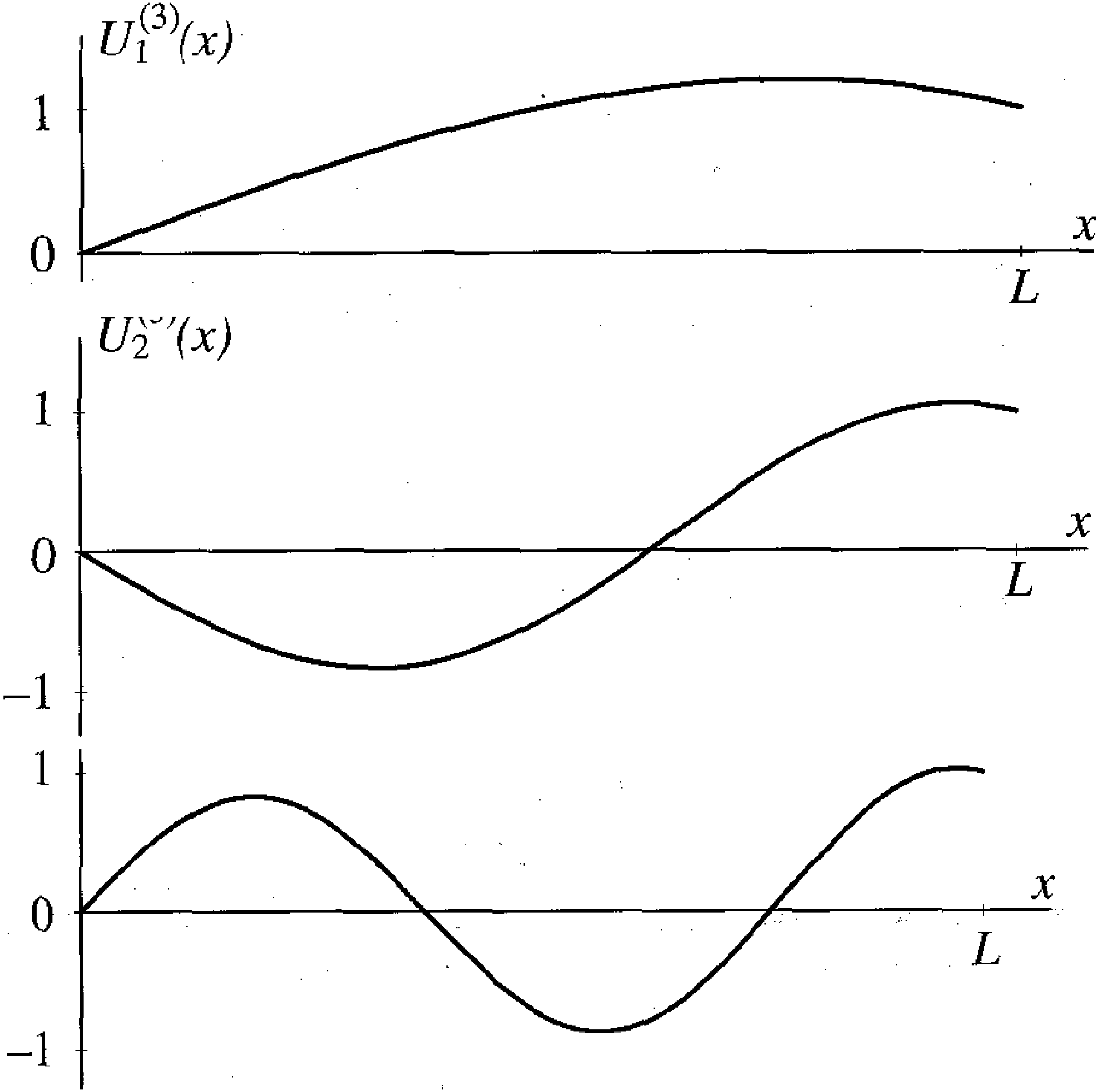
$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} 2.783074 & 0.836697 & -0.247107 \\ 0.836697 & 13.223631 & 2.623716 \\ -0.247107 & 2.623716 & 33.078693 \end{bmatrix}$$

$$M^{(3)} = mL\begin{bmatrix} 0.563196 & 0.085462 & -0.020523 \\ 0.085462 & 0.513392 & 0.070501 \\ -0.020523 & 0.070501 & 0.505321 \end{bmatrix}$$

$$\omega_1^{(3)} = 2.215728\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_1^{(3)} = (mL)^{-1/2}\begin{bmatrix} 1.340184 \\ -0.054456 \\ 0.010464 \end{bmatrix}$$

$$\omega_2^{(3)} = 5.100701\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_2^{(3)} = (mL)^{-1/2}\begin{bmatrix} -0.1617149 \\ 1.419516 \\ -0.053821 \end{bmatrix}$$

$$\omega_3^{(3)} = 8.124264\sqrt{\frac{EA}{mL^2}}, \quad \mathbf{a}_3^{(3)} = (mL)^{-1/2}\begin{bmatrix} 0.067503 \\ -0.155385 \\ 1.422089 \end{bmatrix}$$



$$\omega_1 = 2.2157\sqrt{\frac{EA}{mL^2}}$$

$$\omega_2 = 5.1007\sqrt{\frac{EA}{mL^2}}$$

$$\omega_3 = 8.1243\sqrt{\frac{EA}{mL^2}}$$

Example: Using Comparison Function

Convergence to six decimal places is reached by the three lowest natural frequencies as follows:

$$\omega_1^{(14)} = 2.215524\sqrt{EA/mL^2},$$

$$\omega_2^{(14)} = 5.099525\sqrt{EA/mL^2},$$

$$\omega_3^{(20)} = 8.116318\sqrt{EA/mL^2}$$

AN ENHANCED RAYLEIGH-RITZ METHOD

Improving accuracy, and hence convergence rate, by combining admissible functions from several families,

- each family possessing different dynamic characteristics of the system under consideration

$$U(x) = a_1 \sin \frac{\pi x}{2L} + a_2 \sin \frac{\pi x}{L}$$

Free end

Fixed end

AN ENHANCED RAYLEIGH-RITZ METHOD

The linear combination can be made to satisfy the boundary condition for a spring-supported end

$$EA(L)\left(a_1\frac{\pi}{2L}\cos\frac{\pi x}{2L}+a_2\frac{\pi}{L}\cos\frac{\pi x}{L}\right)\Big|_{x=L}+k\left(a_1\sin\frac{\pi x}{2L}+a_2\sin\frac{\pi x}{L}\right)\Big|_{x=L}$$

$$=-EA(L)a_2\frac{\pi}{L}+ka_1=0$$

$$a_2=\frac{kL}{\pi EA(L)}a_1$$

$$U(x)=a_1\left[\sin\frac{\pi x}{2L}+\frac{kL}{\pi EA(L)}\sin\frac{\pi x}{L}\right]$$

AN ENHANCED RAYLEIGH-RITZ METHOD

Example: Use the given comparison function given in conjunction with Rayleigh's energy method to estimate the lowest natural frequency of the rod of previous example.

$$U(x) = \sin\frac{\pi x}{2L} + \frac{kL}{\pi EA(L)}\sin\frac{\pi x}{L}$$

$$k = EA/L.\ EA(L) = 0.6\ EA$$

$$R(U(x)) = \omega^2 = \frac{V_{\max}}{T_{\mathrm{ref}}}$$

AN ENHANCED RAYLEIGH-RITZ METHOD

$$V_{\max} = \frac{1}{2}\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx + \frac{1}{2}kU^2(L)$$

$$= \frac{1}{2}\left\{\frac{6EA}{5}\int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\left(\frac{\pi}{2L}\cos\frac{\pi x}{2L} + 0.530516\frac{\pi}{L}\cos\frac{\pi x}{L}\right)^2 dx + k\right\}$$

$$= \frac{1}{2}\left\{\frac{6EA}{5}\int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\left[\left(\frac{\pi}{2L}\right)^2\cos^2\frac{\pi x}{2L}\right.\right.$$

$$\left.\left. + 2\times 0.530516\frac{\pi}{2L}\frac{\pi}{L}\cos\frac{\pi x}{2L}\cos\frac{\pi x}{L} + 0.530516^2\left(\frac{\pi}{L}\right)^2\cos^2\frac{\pi x}{L}\right]dx + \frac{EA}{L}\right\}$$

$$= \frac{1}{2}(2.383701 + 2\times 0.530516\times 1.363968 + 0.530516^2\times 4.784802)\frac{EA}{L}$$

$$= \frac{1}{2}\times 5.177584\frac{EA}{L}$$

AN ENHANCED RAYLEIGH-RITZ METHOD

$$T_{\text{ref}} = \frac{1}{2}\int_0^L m(x)U^2(x)dx = \frac{1}{2}\frac{6m}{5}\int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\left(\sin\frac{\pi x}{2L}+0.530516\sin\frac{\pi x}{L}\right)^2 dx$$

$$= \frac{1}{2}\frac{6m}{5}\int_0^L \left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\left(\sin^2\frac{\pi x}{2L}+2\times 0.530516\sin\frac{\pi x}{2L}\sin\frac{\pi x}{L}\right.$$

$$\left. +0.530516^2\sin^2\frac{\pi x}{L}\right)dx$$

$$= \frac{1}{2}(0.439207+2\times 0.530516\times 0.415189+0.530516^2\times 0.515198)$$

$$= \frac{1}{2}\times 1.024737mL \Longrightarrow \omega = \sqrt{\frac{5.177584}{1.024737}\frac{EA}{mL^2}} = 2.247798\sqrt{\frac{EA}{mL^2}}$$

$$\frac{\omega-\omega_1^{(14)}}{\omega} = \frac{2.247798-2.215524}{2.247798} = 0.014358 \cong 1.4\%$$

AN ENHANCED RAYLEIGH-RITZ METHOD

- It is better to regard a1 and a2 as independent undetermined coefficients, and let the Rayleigh- Ritz process determine these coefficients.
- This motivates us to create a new class of functions referred to as quasi-comparison functions
  - ✓ defined as linear combinations of admissible functions capable of satisfying all the boundary conditions *of* the problem

$$U(x) = a_1 \sin \frac{\pi x}{2L} + a_2 \sin \frac{\pi x}{L} + a_3 \sin \frac{3\pi x}{2L} + \ldots + a_n \sin \frac{n\pi x}{2L}$$

$$= \sum_{i=1}^{n} a_i \sin \frac{i\pi x}{2L}$$

AN ENHANCED RAYLEIGH-RITZ METHOD

- One word of caution is in order:
  - Each of the two sets of admissible functions is complete
    - As a result, a given function in one set can be expanded in terms of the functions in the other set.
      - The implication is that, as the number of terms n increases, the two sets tend to become dependent.
      - When this happens, the mass and stiffness matrices tend to become singular and the eigensolutions meaningless.
  - But, because convergence to the lower modes tends to be so fast, in general the singularity problem does not have the chance to materialize.

AN ENHANCED RAYLEIGH-RITZ METHOD

Solve the problem of privious example using the quasi-comparison functions

$$U^{(n)}(x) = \sum_{i=1}^{n} a_i \phi_i(x) = \sum_{i=1}^{n} a_i \sin i\pi x/2L, \quad n = 2, 3, \ldots$$

$$k_{ij}^{(n)} = \int_0^L EA(x) \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx} dx + k\phi_i(L)\phi_j(L)$$

$$= \frac{6EA}{5} \frac{i\pi}{2L} \frac{j\pi}{2L} \int_0^L \left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] \cos\frac{i\pi x}{2L} \cos\frac{j\pi x}{2L} dx + \frac{EA}{L} \sin\frac{i\pi}{2} \sin\frac{j\pi}{2},$$

$$m_{ij}^{(n)} = \int_0^L m(x)\phi_i(x)\phi_j(x)dx = \frac{6m}{5} \int_0^L \left[1 - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] \sin\frac{i\pi x}{2L} \sin\frac{j\pi x}{2L} dx,$$

Example: n=2

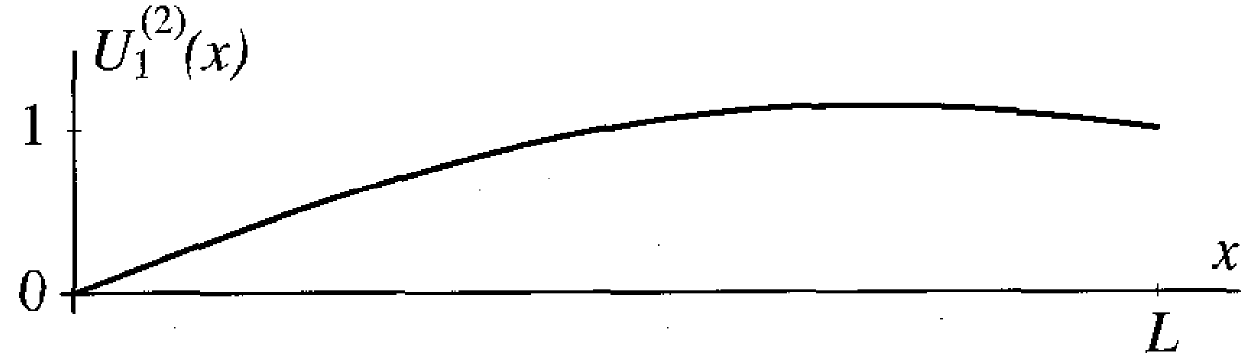
$$K^{(2)} = \frac{EA}{L}\begin{bmatrix} 2.383701 & 1.363968 \\ 1.363968 & 4.784802 \end{bmatrix} M^{(2)} = mL\begin{bmatrix} 0.439207 & 0.415189 \\ 0.415189 & 0.515198 \end{bmatrix}$$

$$\omega_1^{(2)} = 2.223595\sqrt{\frac{EA}{mL^2}}, \ \mathbf{a}_1^{(2)} = (mL)^{-1/2}\begin{bmatrix} 1.159578 \\ 0.357015 \end{bmatrix}$$

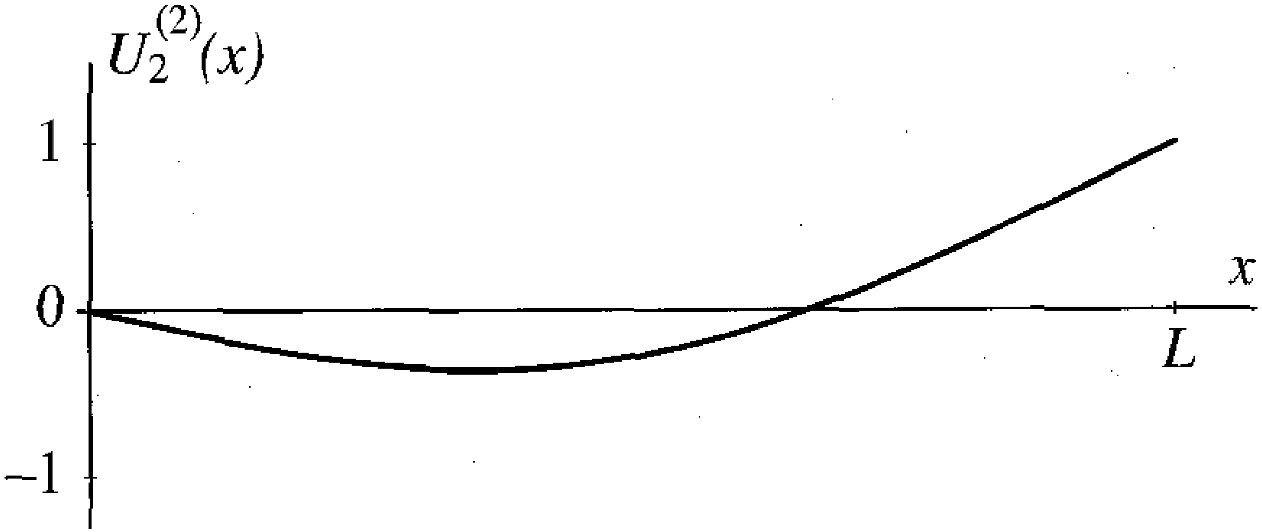
$$\omega_2^{(2)} = 5.984845\sqrt{\frac{EA}{mL^2}}, \ \mathbf{a}_2^{(2)} = (mL)^{-1/2}\begin{bmatrix} 2.866064 \\ -2.832235 \end{bmatrix}$$

$$U_1^{(2)}(x) = 1.159578\sin\frac{\pi x}{2L} + 0.357015\sin\frac{\pi x}{L}$$

$$U_2^{(2)}(x) = 2.866064\sin\frac{\pi x}{2L} - 2.832235\sin\frac{\pi x}{L}$$



$$\omega_1 = 2.2236\sqrt{\frac{EA}{mL^2}}$$



$$\omega_2 = 5.9848\sqrt{\frac{EA}{mL^2}}$$

Example: n=3

$$K^{(3)} = \frac{EA}{L}\begin{bmatrix} 2.383701 & 1.363968 & -0.662500 \\ 1.363968 & 4.784802 & 5.703086 \\ -0.662500 & 5.703086 & 12.253305 \end{bmatrix}$$

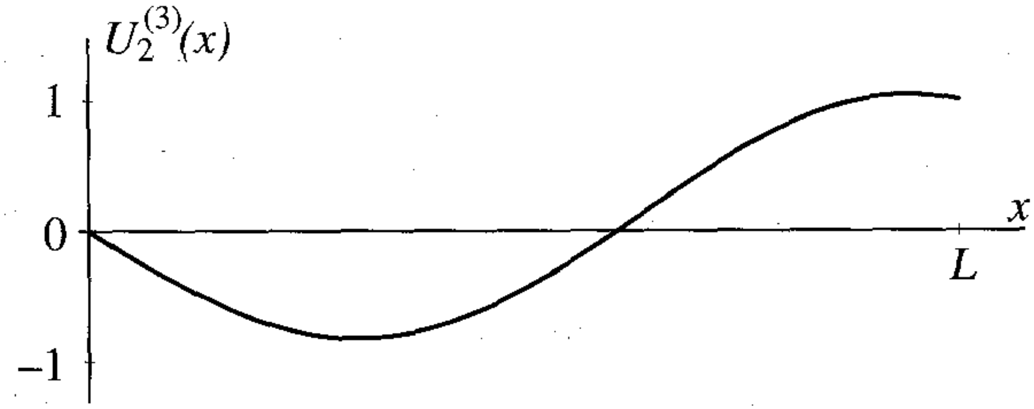
$$M^{(3)} = mL\begin{bmatrix} 0.439207 & 0.415189 & 0.075991 \\ 0.415189 & 0.515198 & 0.306358 \\ 0.075991 & 0.306358 & 0.493245 \end{bmatrix}$$

$$\omega_1^{(3)} = 2.216154\sqrt{\frac{EA}{mL^2}},\ \mathbf{a}_1^{(3)} = (mL)^{-1/2}\begin{bmatrix} 1.028923 \\ 0.519181 \\ -0.113326 \end{bmatrix}$$

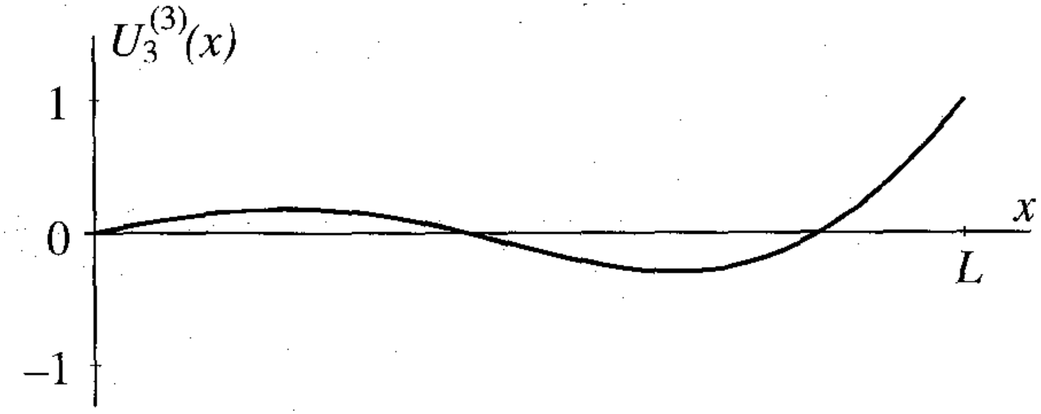
$$\omega_2^{(3)} = 5.100072\sqrt{\frac{EA}{mL^2}},\ \mathbf{a}_2^{(3)} = (mL)^{-1/2}\begin{bmatrix} 0.217568 \\ -0.705970 \\ 1.778731 \end{bmatrix}$$

$$\omega_3^{(3)} = 11.092640\sqrt{\frac{EA}{mL^2}},\ \mathbf{a}_3^{(3)} = (mL)^{-1/2}\begin{bmatrix} -9.597960 \\ 11.040485 \\ -5.308067 \end{bmatrix}$$

$$\omega_1 = 2.2162\sqrt{\frac{EA}{mL^2}}$$

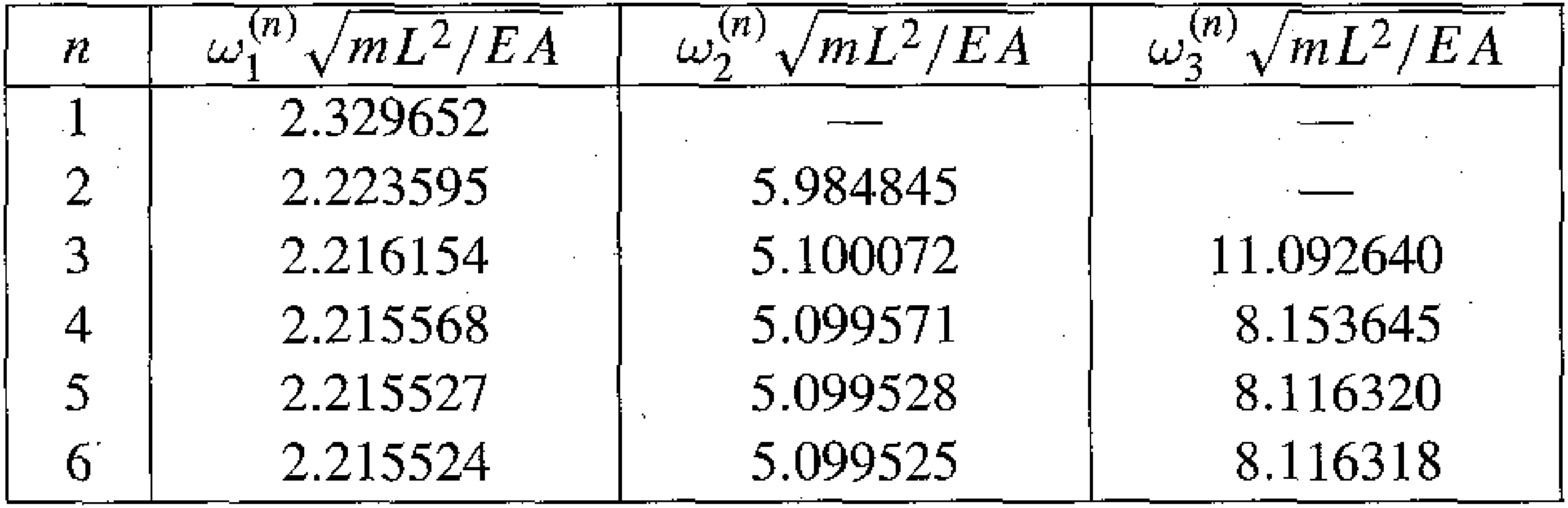


$$\omega_2 = 5.1001\sqrt{\frac{EA}{mL^2}}$$



$$\omega_3 = 11.0926\sqrt{\frac{EA}{mL^2}}$$

AN ENHANCED RAYLEIGH-RITZ METHOD



n	$\omega_1^{(n)}\sqrt{mL^2/EA}$	$\omega_2^{(n)}\sqrt{mL^2/EA}$	$\omega_3^{(n)}\sqrt{mL^2/EA}$
1	2.329652	—	—
2	2.223595	5.984845	—
3	2.216154	5.100072	11.092640
4	2.215568	5.099571	8.153645
5	2.215527	5.099528	8.116320
6	2.215524	5.099525	8.116318

Distributed-Parameter Systems: Approximate Methods

- Rayleigh's Principle
- The Rayleigh-Ritz Method
- An Enhanced Rayleigh-Ritz Method
- The Assumed-Modes Method: System Response
- The Galerkin Method
- The Collocation Method