Types of Applications of Measurement Instrumentation

- WHY STUDY MEASUREMENT SYSTEMS?
- CLASSIFICATION OF TYPES OF MEASUREMENT APPLICATIONS
- COMPUTER-AIDED MACHINES AND PROCESSES
WHY STUDY MEASUREMENT SYSTEMS?

- “Measurement system” includes all components in a chain of hardware and software that leads from the measured variable to processed data.

- Let’s introduce some basic ideas using the automotive industry as an example.

- Modern automobile uses as many as 40 or 50 sensors (measuring devices) in implementing various functions.
WHY STUDY MEASUREMENT SYSTEMS?

• In **design** stage, one must be aware of the instruments available
  ▫ for the various measurements, and
  ▫ how they operate and interface with other parts of the system.
  ▫ keep up with new sensor developments to allow improvements in car design and operation.

• Lack of such sensor knowledge can severely restrict the range of designs that one can conceive, thus limiting improvements in overall car performance.
WHY STUDY MEASUREMENT SYSTEMS?

- **Laboratory testing** and the associated measurement systems are a *vital part of the design process*:
  - If a new material is being considered, we may need to run strength tests to develop data needed by the design engineers.
  - Or, a new or revised manufacturing process may require statistical *response surface experiments* to find the effects of process variables on performance and/or cost.
  - Finally, availability from suppliers of new components, such as improved shock absorbers, may require performance testing to decide whether their use is warranted in the new design.
WHY STUDY MEASUREMENT SYSTEMS?

• As design and development proceed, prototype subsystems and finally entire vehicles will be produced.
• These are used as "test beds" to evaluate performance and then feed back information to the design/manufacturing teams.
• Once the design has been finalized, then manufacture of the product in quantity can commence,
  ▫ the manufacturing tools are controlled by a so-called feedback mechanism
  ▫ some quality parameter of the part produced is measured with appropriate sensors.
WHY STUDY MEASUREMENT SYSTEMS?

The final product, a modern automobile, relies on a multitude of sensors for its optimum operation:

- Speedometers tell us the vehicle's speed
- Tachometers display engine RPM
- Fuel gages keep track of the gas supply
- Temperature sensors warn of overheating
- GPS to locate the car and guide the driver
- Accelerometers signal air bags to deploy in case of a crash
- Brake-cylinder pressure and wheel-speed sensors control the antilock braking system
- GyroChip to measure angular velocity to augment vehicle stability during severe or emergency maneuvers.
CLASSIFICATION OF TYPES OF MEASUREMENT APPLICATIONS

• Examples from *any industry, can be classified into only three major categories:*
  ▫ Monitoring of processes and operations  
    • keep track of some quantity  
  ▫ Control of processes and operations  
  ▫ Experimental engineering analysis.

• Every application of measurement can be put into one of these three groups or some combination of them.
General Concepts:

• **Chapter 2: Generalized Configurations and Functional Descriptions of Measuring Instruments**
  - Functional Elements of an Instrument
  - Active and Passive Transducers
  - Analog and Digital Modes of Operation
  - Null and Deflection Methods
  - Input-Output Configuration of Instruments and Measurement Systems
  - Methods of Correction for Interfering and Modifying Inputs

• **Chapter 3: Generalized Performance Characteristics of Instruments**
  - Static Characteristics and Static Calibration
  - Dynamic Characteristics
Measuring Devices

- **Chapter 4**: Motion and Dimensional Measurement
- **Chapter 5**: Force, Torque, and Shaft Power Measurement
- **Chapter 6**: Pressure and Sound Measurement
- **Chapter 7**: Flow Measurement
- **Chapter 8**: Temperature and Heat-Flux Measurement
Fundamentals of Signal Analysis

- Time, and Frequency Domains
  - The Time Domain
  - The Frequency Domain
  - Instrumentation for the Frequency Domain
Fundamentals of Signal Analysis

- Understanding Dynamic Signal Analysis
  - FFT Properties
  - Sampling and Digitizing
  - Aliasing
  - Band Selectable Analysis
  - Windowing
  - Averaging
  - Real Time Bandwidth
  - Overlap Processing

- Using Dynamic Signal Analyzers
References & Course Evaluation

- **Title:** Measurement systems: application and design
- **By:** Ernest O. Doebelein. 5th ed.
- **Publisher:** McGraw-Hill series in mechanical and industrial engineering

**Course Evaluation Scheme:**
- Mid-Term 25% (chapters 2-3)
- Lab Reports 35% (3 selected experiments)
- Final Exam 40%
Lecture 2- Generalized Configurations and Functional Descriptions of Measuring Instruments

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Generalized Configurations and Functional Descriptions of Measuring Instruments

- Functional Elements of an Instrument
- Active and Passive Transducers
- Analog and Digital Modes of Operation
- Null and Deflection Methods
- Input-Output Configuration of Instruments and Measurement Systems
  - Methods of Correction for Interfering and Modifying Inputs
Functional Elements of an Instrument

• In general without recourse to specific physical hardware, one may describe both the operation and the performance of measuring instruments,
  ▫ The operation can be described in terms of the functional elements of instrument systems,
    • such scheme helps to understand the operation of any new instrument with which one may come in contact and to plan the design of a new instrument.
  ▫ The performance is defined in terms of the static and dynamic performance characteristics.
Functional Elements of an Instrument

Simple Instrument Model

Key functional element

Mass of an object

Weight

(Measurable Physical Variable)

Mechanical or Electrical

(Can be Manipulated in a Transmission System)

The Measurement

(Observed Output)
Functional Elements of an Instrument

- A possible arrangement that includes all the **basic functions**,  
  - It is a vehicle for presenting the concept of functional elements, and not as a physical schematic of a generalized instrument.

  e.g. mechanical/electronic amplifier
Functional Elements of an Instrument
Pressure gage

- The primary sensing element is the piston, which also serves the function of variable conversion element.
Functional Elements of an Instrument
Pressure thermometer

- The liquid-filled bulb acts as a primary sensor and variable-conversion element
  - A temperature change results in a pressure buildup within the bulb
Active and Passive Transducers

• One may group the instruments based on energy considerations:
  ▫ a physical component may act as an active transducer or a passive transducer.

• Passive transducer: A component whose output energy is supplied entirely by its input signal

• An active transducer has an auxiliary source of power which supplies a major part of the output power while the input signal supplies only an insignificant portion.
Active and Passive Transducers

- **Passive**: Do not add energy but may remove (thermocouple,...)

- **Active**: Add energy to the measuring environment (radar,...)
Active and Passive Transducers

- The electronic amplifier:
  - the input-signal voltage, $e_i$, need supply only a negligible amount of power
  - almost no current is drawn, owing to negligible gate current and a high $R_g$
  - the output element (the load resistance $R_L$) receives significant current and voltage and thus power.
  - power must be supplied by the battery $E_{bb}$
  - the input controls the output, but does not actually supply the output power.
Analog and Digital Modes of Operation

- The majority of primary sensing elements are of the analog type.
  - For analog signals, the precise value of the quantity (voltage, rotation angle, etc.) carrying the information is significant.
- Digital signals are basically of a binary (on/off) nature, and variations in numerical value are associated with changes in the logical state ("true/false") of some combination of "switches."
  - The system is quite tolerant of spurious "noise" voltages which might contaminate the information signal.
Null and Deflection Methods

- Deflection Instrument

- For either static or dynamic measurements
- High dynamic response
- Energy drain from the measured ... **loading error**
Null and Deflection Methods

- **Null Instrument**
  - Key features
    - Comparator for Iterative balancing operation
    - Feedback to achieve balance
    - Null deflection at parity
    - High accuracy for small input values
    - Low loading error
    - Not suitable for high speed measurements
Measurement Systems

Lecture 3- Input-Output Configuration of Instruments and Measurement Systems

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INPUT-OUTPUT CONFIGURATION OF INSTRUMENTS AND MEASUREMENT SYSTEMS

- A generalized configuration containing the significant input-output relationships present in all measuring apparatus,
  - A scheme suggested by Draper, McKay, and Lees
- Desired inputs: quantities that the instrument is specifically intended to measure.
- Interfering inputs: quantities to which the instrument is unintentionally sensitive.
- Modifying inputs are the quantities that cause a change in the input-output relations for the desired and interfering inputs
Examples: Interfering/Modifying inputs

The desired inputs $p_1$ and $p_2$ whose difference causes the output $x$, which can be read off the calibrated scale.

Measuring pressures under acceleration influence; an error will be engendered because of the interfering acceleration input.

Modifying inputs: ambient temperature and gravitational force.

Both the desired and the interfering inputs may be altered by the modifying inputs.
Examples: Interfering/Modifying inputs

- **Interfering inputs:**
  - The 60-Hz magnetic field induces voltages in the strain-gage circuit.
  - The gage temperature causes a change in gage resistance; cause a voltage output even if there is no strain.
  - Temperature causes a differential expansion of the gage which gives rise to a strain.

- **Modifying input:**
  - The gage factor is sensitive to temperature
  - The battery voltage $E_b$ also changes the proportionality factor

\[
\Delta R_g = (GF)R_g \varepsilon
\]
\[
\Delta R_g \triangleq \text{change in gage resistance}, \ \Omega
\]
\[
GF \triangleq \text{gage factor, dimensionless}
\]
\[
R_g \triangleq \text{gage resistance when unstrained}, \ \Omega
\]
\[
\varepsilon \triangleq \text{unit strain, cm/cm}
\]
\[
e_o = -(GF)R_g \varepsilon E_b \frac{R_a}{(R_g + R_a)^2}
\]
Methods of Correction for Interfering and Modifying Inputs

- A number of methods for nullifying/reducing the effects of spurious inputs are available:
  - The *method of inherent insensitivity*
  - The *method of high-gain feedback*
  - The *method of calculated output corrections*
  - The *method of signal filtering*
  - The *method of opposing inputs*
The method of inherent insensitivity

- The elements of the instrument should inherently be sensitive to only the desired inputs:
  - Choosing gage material that exhibits an extremely low temperature coefficient of resistance while retaining its sensitivity to strain.
  - In mechanical apparatus that must maintain accurate dimensions in the face of ambient-temperature changes, the use of a material of very small temperature coefficient of expansion may be helpful.
The method of high-gain feedback

- E.g.: Measuring a voltage $e_i$ by applying it to a motor whose torque acts on a spring, causing a displacement $x_o$

$$x_o = (K_{Mo} K_{SP}) e_i$$

Open-loop system
The method of high-gain feedback

- The output $x_o$ is measured by the feedback device, which produces a voltage $e_o$ proportional to $x_o$

- We now require $K_{FB}$ stay constant (unaffected by $i_{M4}$)

$$x_o = \frac{K_{AM} K_{Mo} K_{SP}}{1 + K_{AM} K_{Mo} K_{SP} K_{FB}} e_i$$

$K_{AM}$ to be very large ("high-gain")

$$K_{AM} K_{Mo} K_{SP} K_{FB} \gg 1 \quad \Rightarrow \quad x_o \approx \frac{1}{K_{FB}} e_i$$
The method of calculated output corrections

- Requires to measure or estimate the magnitudes of the interfering and/or modifying inputs and to know quantitatively how they affect the output:
  - In the manometer the effects of temperature on both the calibrated scale's length and the density of mercury may be quite accurately computed.
  - The local gravitational acceleration is known for a given elevation and latitude, so that this effect may be corrected.
The method of signal filtering

Input filtering

Output filtering
The method of input signal filtering

- Electromechanical devices for navigation and control of aircraft or missiles,
  - The interfering vibration input may be filtered out by use of suitable spring mounts.

- The interfering tilt-angle input to the manometer may be effectively filtered out by means of the gimbal-mounting
The method of input signal filtering

- The thermocouple reference junction is shielded from ambient temperature fluctuations.
- Such an arrangement acts as a filter for temperature or heat-flow inputs.
- The strain-gage circuit is shielded from the interfering 60-Hz field.
The method of output signal filtering

- The strains to be measured are mainly steady and never vary more rapidly than 2 Hz.
- It is possible to insert a simple RC filter that will pass the desired signals but almost completely block the 60-Hz interference.
The method of output signal filtering

- The pressure gage modified by the insertion of a flow restriction between the source of pressure and the piston chamber,
  - The pulsations in the air pressure may be smoothed by the pneumatic filtering effect of the flow restriction and associated volume.
The method of output signal filtering

- A "chopped" radiometer senses the temperature $T_s$ in terms of the infrared radiant energy emitted.
- The difficulty is that the ambient temperature, as well as $T_s$, affects $T_d$.
- Interposing a rotating shutter between the radiant source and the detector, so that the desired input is "chopped," or modulated, at a known frequency.
The method of opposing inputs

- Intentionally introducing into the instrument interfering and/or modifying inputs that tend to cancel the bad effects of the unavoidable spurious inputs.
The method of opposing inputs

- A millivoltmeter is basically a current-sensitive device.
- However, as long as the total circuit resistance is constant, its scale can be calibrated in voltage, since voltage and current are proportional.
The method of opposing inputs

- This velocity increase due to change in streamline causes a drop in static pressure.
- By properly choosing distances $d_1$ and $d_2$, two under/overpressure effects can be made exactly to cancel, giving a true static-pressure value at the tap.
The method of opposing inputs

- The mass flow rate of gas through an orifice may be found by measuring the pressure drop across the orifice.
- Variations in gas temperature and pressure yield different mass flow rates for the same orifice pressure drop.
- Opposing input is accomplished by attaching the specially shaped metering pin to a gas-filled bellows, enabling the flow area to be varied in just the right way.
The method of opposing inputs

- The action of the device is that a vehicle rotation at angular velocity $\theta_i$ causes a proportional displacement $\theta_o$ of the gimbal relative to the case.
- When the temperature increases, viscosity drops, causing a loss of damping.
- Simultaneously, the nylon cylinder expands, narrowing the damping gap and thus restoring the damping to its proper value.
Measurement Systems

Lecture 4- Generalized Performance Characteristics of Instruments

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INTRODUCTION

• Study the performance of measuring instruments and systems with regard to:
  ▫ how well they measure the desired inputs, and
  ▫ how thoroughly they reject the spurious inputs.

• The treatment of instrument performance characteristics is broken down into the subareas of
  ▫ static characteristics, and
  ▫ dynamic characteristics.

• The overall performance of an instrument is then judged by a semi-quantitative superposition of the static and dynamic characteristics.
Static Characteristics and Static Calibration

- Meaning of Static Calibration
- Measured Value versus True Value
- Some Basic Statistics
- Least-Squares Calibration Curves
- Calibration Accuracy versus Installed Accuracy
- Combination of Component Errors in Overall System-Accuracy Calculations
- Theory Validation by Experimental Testing
- Effect of Measurement Error on Quality Control Decisions in Manufacturing
- Static Sensitivity
- Computer-Aided Calibration and Measurement: Multiple Regression
- Linearity
- Threshold, Noise Floor, Resolution, Hysteresis, and Dead Space
- Scale Readability
- Span
- Generalized Static Stiffness and Input Impedance: Loading Effects
- Concluding Remarks on Static Characteristics
Meaning of Static Calibration

- Static calibration:
  - all inputs (desired, interfering, modifying) except one are kept constant.
  - the input-output relations is developed

- Superposition of these individual effects describes the overall instrument static behavior.

- The calibration system should have a total uncertainty four times better than the unit under test.
Meaning of Static Calibration

• In performing a calibration, the following steps are necessary:
  ▫ Examine the construction of the instrument, and identify and list all the possible inputs.
  ▫ Decide, as best you can, which of the inputs will be significant in the application for which the instrument is to be calibrated.
  ▫ Procure apparatus that will allow you to vary all the significant inputs over the ranges considered necessary. Procure standards to measure each input.
  ▫ By holding some inputs constant, varying others, and recording the output(s), develop the desired static input-output relations.
The term "true value" refers to a reference value that would be obtained if the quantity under consideration were measured by an exemplar method; a method agreed on by experts as being sufficiently accurate for the purposes to which the data ultimately will be put.

If measurement process is repeated under assumed identical conditions, we get a number of readings from the instrument; the process must be in a state of statistical control.
Measured Value versus True Value

- Every instrument has an infinite number of inputs;
- In a calibration procedure, certain inputs are held "constant" within certain limits.
- These inputs contribute the largest components to the overall error of the instrument.
- The remaining infinite number of inputs is left uncontrolled, and it is hoped that each of these individually contributes only a very small effect.
- In the aggregate these effect on the instrument output will be of a random nature.
Measured Value versus True Value

- Effect of uncontrolled input on calibration
- A measurement process with good statistical control generates a set of data exhibiting random scatter.

Calibration was carried out without temperature control. Performing the calibration in a temperature-controlled room.
Some Basic Statistics

- Pressure-gage calibration data

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<th>Trial number</th>
<th>Scale reading, kPa</th>
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<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>10.20</td>
</tr>
<tr>
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<td>19</td>
<td>10.04</td>
</tr>
<tr>
<td>20</td>
<td>9.81</td>
</tr>
</tbody>
</table>

![Histogram of scale readings with intervals from 9.80 to 10.50 kPa]
Some Basic Statistics

- **Histogram presentation:**

\[ Z \triangleq \frac{(\text{number of readings in an interval})/\text{(total number of readings)}}{\text{width of interval}} \]

- The area of a particular "bar" is numerically equal to the *probability that a specific reading will fall in the associated interval*.
- The area of the entire histogram must then be 1.0.
Some Basic Statistics

• In limit with infinite number of readings, each with an infinite number of significant digits, \( Z = f(x) \) is called the \textit{probability density function}.

Probability of reading lying between \( a \) and \( b \)

\[
P(a < x < b) = \int_{a}^{b} f(x) \, dx
\]

The cumulative distribution function

\[
F(x) = \int_{-\infty}^{x} f(x) \, dx
\]
The normal or Gaussian PDF

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$-\infty < x < +\infty$
Normal (Gaussian) Distribution

Random variable

Non-dimensional variable

the population mean

\[ z = \frac{x - \mu}{\sigma} \]

\[ f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \]
Clearance deviations

- Consider a shaft in a bearing $D_S=25.400$ mm, and $D_B=25.451$ mm. The standard deviation of the shaft diameter is $0.008$ mm, and the standard deviation of the bearing diameter is $0.010$ mm. For satisfactory operation the difference in diameters (clearance) between the bearings must be between $0.0381$ mm and $0.0635$ mm. What fraction of the final assemblies will be rejected?

\[ C = 25.451 - 25.400 = 0.051 \, \text{mm} \]

\[ S_c = \sqrt{0.008^2 + 0.010^2} = 0.0128 \, \text{mm} \]

\[ \frac{0.0635 - C}{S_c} = 0.98, \quad \frac{0.0381 - C}{S_c} = -1.01 \]
Clearance deviations

Considering C follows a Gaussian distribution 68.2% of products are accepted. The remaining are rejected.
Qualitative test for conformity to the Gaussian distribution

To plot Gaussian line, one must estimate

\[ \overline{X} \triangleq \frac{\sum_{i=1}^{N} X_i}{N} \quad s \triangleq \sqrt{\frac{\sum_{i=1}^{N} (X_i - \overline{X})^2}{N - 1}} \]

\( X_i \triangleq \text{individual reading} \)
\( N \triangleq \text{total number of readings} \)

68% of the readings lie within ± 1σ of μ
95% of the readings lie within ± 2σ of μ
99.7% of the readings lie within ± 3σ of μ
The effect of sample size increasing

- When real distribution is nearly Gaussian, if more readings were taken, 99.7 percent would fall within $10.11 \pm 0.42$ kPa
Display an empirical quantile-quantile plot

```matlab
%% Measurement systems
%% Gage calibration example
%( simulated using normally distributed pseudorandom numbers)
%
clc
close all
X=10+0.14*randn(100,1); % mean=1, STD=2
figure(1), hist(X,10)
figure(2), qqplot(X)
```
Student's t Distribution

• Continuous, symmetrical distribution, used for analysis of the variation of sample mean value for experimental data with sample size less than 30.
  ▫ If the sample size is small \((n < 30)\), the assumption that the population standard deviation can be represented by the sample standard deviation may not be accurate.
  ▫ Due to the uncertainty in the standard deviation, for the same confidence level, we would expect the confidence interval to be wider.

• For sample sizes greater than 30, Student's t approaches normal distribution.
Student's t Distribution

- In case of small samples, a statistic called Student's t is used:

\[ t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \]

- the sample mean
- the sample size
- the sample standard deviation
- degrees of freedom

Gamma function

\[ f(t, \nu) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\sqrt{\nu \pi} \Gamma\left(\frac{\nu}{2}\right) \left(1 + \frac{t^2}{\nu}\right)^{(\nu+1)/2}} \]
Student's t Distribution

- Like the normal distribution, these are symmetric curves.
- As the number of samples increases, the t-distribution approaches the normal distribution.
- For lesser values of $\nu$, the distribution is broader with a lower peak.

$$f(t, \nu) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\sqrt{\nu \pi} \Gamma\left(\frac{\nu}{2}\right) \left(1 + \frac{t^2}{\nu}\right)^{(\nu+1)/2}}$$
Confidence interval for the t-distribution

\[ t = \frac{\bar{x} - \mu}{S/\sqrt{n}} \]

The probability that \( t \) lies in the confidence interval, between \(-t_{\alpha/2}\) and \( t_{\alpha/2} \):

\[ P[-t_{\alpha/2} \leq t \leq t_{\alpha/2}] = 1 - \alpha \]

\[ P\left[ \bar{x} - t_{\alpha/2} \frac{S}{\sqrt{n}} \leq \mu \leq \bar{x} + t_{\alpha/2} \frac{S}{\sqrt{n}} \right] = 1 - \alpha \]

\[ \mu = \bar{x} \pm t_{\alpha/2} \frac{S}{\sqrt{n}} \text{ with confidence level } 1 - \alpha \]

For example, for a 95% confidence level, \( \alpha = 1 - 0.95 = 0.05 \) and \( \alpha/2 = 0.025 \).
Student's \( t \) as a Function of \( \alpha \) and \( v \)

\[
\begin{array}{cccccc}
\alpha/2 & 0.010 & 0.005 & 0.025 & 0.010 & 0.005 \\
\hline
0.100 & 3.078 & 6.314 & 12.706 & 31.823 & 63.658 \\
0.050 & 1.886 & 2.920 & 4.303 & 6.964 & 9.925 \\
0.025 & 1.638 & 2.353 & 3.182 & 4.541 & 5.841 \\
0.010 & 1.533 & 2.132 & 2.776 & 3.747 & 4.604 \\
0.005 & 1.476 & 2.015 & 2.571 & 3.365 & 4.032 \\
\end{array}
\]
Example:

- A manufacturer would like to estimate the mean failure time of its products with 95% confidence. Six systems are tested to failure, and the following data (in hours of functioning time) are obtained: 1250, 1320, 1542, 1464, 1275, and 1383. Estimate the population mean and the 95% confidence interval on the mean.

\[
\bar{x} = \frac{1250 + 1320 + 1542 + 1464 + 1275 + 1383}{6} = 1372 \text{ h}
\]

\[
S = \left( \frac{(1250 - 1372)^2 + (1320 - 1372)^2 + (1542 - 1372)^2 + (1464 - 1372)^2 + (1275 - 1372)^2 + (1383 - 1372)^2}{5} \right)^{1/2} = 114 \text{ h}
\]

\[
\mu = \bar{x} \pm t_{\alpha/2} \frac{S}{\sqrt{n}} = 1372 \pm 2.571 \times \frac{114}{\sqrt{6}} = 1372 \pm 120 \text{ h}
\]

If we were to increase the confidence level, the estimated interval will also expand, and vice versa.
Interval Estimation of the Population Variance

- In many situations, the variability of the random variable is as important as its mean value.
- The best estimate of the population variance, $\sigma^2$, is the sample variance, $S^2$.
- .......
Measurement Systems

Lecture 5-Least-Squares Calibration Curves

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Least-Squares Calibration Curves

- In instrument calibration, the true value is varied, in increments, over some range, causing the measured value also to vary over a range.
- The procedure is merely to cover the desired range in both the increasing and the decreasing directions.
- The *average calibration curve* for an instrument generally is taken as a straight line which fits the scattered data points best.
- An example of pressure gage calibration.
Least-Squares Calibration Curves

\[ q_o = mq_i + b \]

\[
m = \frac{N \Sigma q_i q_o - (\Sigma q_i)(\Sigma q_o)}{N \Sigma q_i^2 - (\Sigma q_i)^2}
\]

\[
b = \frac{(\Sigma q_o)(\Sigma q_i^2) - (\Sigma q_i q_o)(\Sigma q_i)}{N \Sigma q_i^2 - (\Sigma q_i)^2}
\]

\[
\begin{bmatrix}
q_{o1} \\
q_{o2} \\
\vdots \\
q_{on}
\end{bmatrix}
= 
\begin{bmatrix}
q_{i1} & 1 \\
q_{i2} & 1 \\
\vdots & \vdots \\
q_{in} & 1
\end{bmatrix}
\begin{bmatrix}
m \\
b
\end{bmatrix}
\]

\[ y = Ax, \quad x = A^+ y \]

\[ A^+ = (A^T A)^{-1} A^T \]
Least-Squares Calibration Curves

\[ x = 1.0823 - 0.8470 \]
Least-Squares Calibration Curves

• The model parameters are derived from scattered data; it would be useful to have some idea of their possible variation:
  • The standard deviation of $q_o$,
    \[ s_{q_o}^2 = \frac{1}{N-2} \sum (mq_i + b - q_o)^2 \]
    □ if $q_i$ were fixed and then repeated measurement of $q_o$ would give scattered values,
  • The standard deviations of $m$ and $b$ may be found from:
    \[ s_m^2 = \frac{Ns_{q_o}^2}{N\sum q_i^2 - (\Sigma q_i)^2} \]
    \[ s_b^2 = \frac{s_{q_o}^2 \sum q_i^2}{N\sum q_i^2 - (\Sigma q_i)^2} \]
Least-Squares Calibration Curves

- Assume $s_{q_o}$ would be the same for any value of $q_i$, $S_{q_o} = 0.208$ kPa, $S_m = 0.0140$ and $S_b = 0.0830$ kPa
- Assuming a Gaussian distribution and the 99.7 percent limits ($\pm 3s$),
  
  $$m = 1.082 \pm 0.042, b = -0.847 \pm 0.249$$
  
- The least-squares line gives:
  
  $$q_i = \frac{q_o + 0.847}{1.082}$$
  
- The $q_i$ value computed in this way must have some error limits put on it
Least-Squares Calibration Curves

- If we were measuring an unknown pressure using this gage and got a reading of $q_{o}$, our estimate of the true pressure would be:

$$q_{i} \Rightarrow \frac{q_{o} - b}{m} \pm 3S_{q_{i}}$$

$$S_{q_{i}} = \frac{1}{N-2} \sum \left( \frac{q_{o} - b}{m} - q_{i} \right)^{2} = \frac{S_{q_{o}}}{m^{2}}$$

- In case of this gage $s_{q_{i}} = 0.192 \text{ kPa}$ and for $q_{o} = 4.32 \text{ kPa}$, the true pressure would be $4.78 \pm 0.58 \text{ kPa}$.
Least-Squares Calibration Curves

- Calibration allows decomposition of the total error of a measurement process into two parts:
  - The bias, also called the systematic error (since it is the same for each reading and thus can be removed by calibration), and
  - The imprecision, also called the random error since it is, in general, different for every reading and we can only put bounds on it, but cannot remove it.
Uncertainty Calculation using t-Distribution

- **Improvements with two major features:**
  - A simple method considers a standard deviation calculated from a small number of points to be as accurate as one gotten from a large number of points.
  - Statistical theory (confidence intervals) allows us to adjust the uncertainty to suit the number of points.
  - The second improvement substitutes for our 3s limits (99.7 percent level) a limit analogous to 2s (95 percent level).
Uncertainty Calculation using t-Distribution

• For Gaussian and near-Gaussian distributions, 99.7 percent puts us well into the "tails" of the distribution.
  ▫ it takes very large samples to get reliable results for probabilities
• Since most engineering samples are relatively small, quoting uncertainty bands at the 95 percent level of confidence is more realistic
Uncertainty Calculation using t-Distribution

- Mandel gives a ± 95 percent confidence interval, defined by two hyperbolas on either side of the least-squares line.
- The "vertical" location of the two hyperbolas as a function of $q_i$ is computed from:

$$
\Delta q_o = \pm t_{95, N-2} s_q \sqrt{\frac{1}{n} + \frac{1}{N} + \frac{N(q_i - \bar{q}_i)^2}{N\Sigma q_i^2 - (\Sigma q_i)^2}}
$$

Here is a table of t distribution values for uncertainty calculation:

<table>
<thead>
<tr>
<th>Degrees of Freedom</th>
<th>50</th>
<th>90</th>
<th>95</th>
<th>98</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>40</td>
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<tr>
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<td>1.671</td>
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<td>120</td>
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<td>1.980</td>
<td>2.358</td>
</tr>
<tr>
<td>$\infty$</td>
<td>0.674</td>
<td>1.645</td>
<td>1.960</td>
<td>2.326</td>
</tr>
</tbody>
</table>
Uncertainty Calculation using t-Distribution

• For a read out \((q_0)\), we draw a horizontal line through that value.
• This line intersects the two hyperbolas, and the \(q_i\) values at these two intersections define the ends of a 95 percent confidence interval for the true value.
Uncertainty Calculation using t-Distribution

• Visually, the two "hyperbolas" seem to be straight lines, but inspection of the tabular results shows that $\Delta q_o$ does vary with $q_i$
  - the largest value being at the left and right ends of the curves and the smallest being at the center.

$$\Delta q_o = \pm t_{95,N-2} s_{q_o} \sqrt{\frac{1}{n} + \frac{1}{N} + \frac{N(q_i - \bar{q})^2}{N \Sigma q_i^2 - (\Sigma q_i)^2}}$$
Uncertainty Calculation using t-Distribution

- A major assumption of the analysis is that the statistical variability of the measurements is the same over the entire calibration range.
- It is good practice to also make a plot of the residuals versus $q_i$;
  - Such a graph can show whether this assumption is reasonable.
- For our example, shows no obvious trend in the size of the residuals; the variability seems to be about the same over the whole range.
Pressure gage imprecision

- The average calibration curve for a pressure gage is \( q_0 = 1.0823q_i - 0.8470, \quad S_{q_0} = 0.208 \text{kPa} \)
- For \( q_o = 4.320 \text{kPa} \), what is the best estimate of the true pressure?
  \[
  q_i = \frac{q_o + 0.8470}{1.0823} = 4.691 \text{kPa}
  \]
- Calculate the imprecision in the estimation of input pressure with the 95% confidence interval.
  \[
  q_i \pm 2S_{q_i} = 4.691 \pm 2 \frac{0.208}{1.0823} = 4.691 \pm 0.384 \text{kPa}
  \Rightarrow [4.307 - 5.075]
Pressure gage imprecision

$q_0 = 4.320$
Pressure gage imprecision

- A 95 percent confidence interval can be defined by two hyperbolas on either side of the least-squares line where the "vertical" location of the two hyperbolas as a function of \( q_i \) is computed from:

\[
\Delta q_o = \pm t_{95, N-2} \cdot s_{q_o} \sqrt{\frac{1}{n} + \frac{1}{N} + \frac{N(q_i - \bar{q_i})^2}{N\Sigma q_i^2 - (\Sigma q_i)^2}}
\]

- Calculate \( \Delta q_o \) for the pressure gage when the input pressure is 5 kPa. Note: The experiments is repeated twice (\( n=2 \)).

\[
N = 22, \quad \Delta q_o = \pm 2.086 \times 0.208 \times \sqrt{\frac{1}{2} + \frac{1}{22}} + 0. = 0.32 \text{ kPa}
\]

- At which input pressure(s) \( \Delta q_o \) is maximized?

@ 0. & 10 kPa

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<thead>
<tr>
<th>kPa</th>
<th>Increasing</th>
<th>Decreasing</th>
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Pressure gage imprecision
Measurement Systems

Lecture 6- Calibration Accuracy/Overall System-Accuracy Calculations

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Calibration Accuracy versus Installed Accuracy

• It was stated that calibration removes the bias portion of the error,
• This is true only for the conditions under which the calibration was performed:
  ▫ This means that the measurement error (bias and imprecision) must be re-evaluated, taking into account, as best possible, the deviation of the measurement conditions from the calibration conditions.
  ▫ This re-evaluation is usually not as straightforward as the calibration was because the measurement environment is rarely as controlled as a standards laboratory calibration.
Differences between calibration and measurement situations

- A simple spring-type force measuring scale could easily be calibrated with standard masses,
  - find a best-fit line and uncertainty, and remove any scale bias present.
- If the temperatures at calibration/measurement are different, the scale will exhibit an uncorrected bias with two sources:
  - thermal expansion (which shifts the zero point) and
  - temperature sensitivity of the spring's elastic modulus (which changes the spring stiffness).
- Other possible effects include angular misalignment of the unknown force with the scale's sensitive axis.
Calibration Accuracy versus Installed Accuracy

• One aspect of the measurement situation is that the bias portion of the error is now not zero. (Recall that we earlier said that calibration removes the bias.)
• Biases are classified into five different types:
  ▫ **Large known biases** (eliminated by calibration).
  ▫ **Large unknown biases** (not correctable; usually come from human errors in data processing, incorrect installation and/or handling of instrumentation, and unexpected environmental disturbances).
    • In a well-controlled measurement process, the assumption is that there are no large unknown biases.
  ▫ **Small known biases** (may or may not be corrected, depending on the correction difficulty and their magnitude.).
  ▫ **Small unknown biases with unknown algebraic sign**.
  ▫ **Small unknown biases with known algebraic sign**.
Calibration Accuracy versus Installed Accuracy

• Small, unknown biases remain as a contribution to the measurement error.

• The bias in the measurement situation (as contrasted with calibration) is treated as a random effect rather than as systematic
  ▫ **Bias limit**: It is defined as the range of values within which we feel that the actual bias will be found 95 percent of the time.

• Using this scheme, the "total error"/uncertainty in the measurement is the sum of the bias limit and the imprecision,

\[
\text{Uncertainty} \triangleq U \triangleq \pm (B + t_{95, N-1}s) \\
B \triangleq \text{bias limit} \\
S \triangleq \text{sample standard deviation,}
\]
Calibration Accuracy versus Installed Accuracy

• To compute uncertainty in force measuring scale, we need to estimate the temperature, misalignment, and any other effects felt to be significant.

• Note that:
  ▫ **we do not** measure the temperature and misalignment and then **correct for these effects**, rather
  ▫ we estimate some limits on how large we think these effects might be and then **add this to the uncertainty**.
Calibration Accuracy versus Installed Accuracy

• A displacement-measuring dial indicator as part of an experiment to find the beam's spring constant, $F/\delta$.

• A bias error will be introduced because the indicator spring force acts against $F$,
  ▫ Causing the measured deflection to be less than it should be.
  ▫ If $F$ is always downward, this bias error would be treated as having an unknown magnitude, but a known sign.
  ▫ The deflection is always measured too low.

• If we estimate an upper limit for its magnitude, this bias would give an unsymmetrical uncertainty; for example, -0.003 in. to +0.001 in.
Calibration Accuracy versus Installed Accuracy

- Thermocouples are calibrated in an accurately controlled and measured temperature environment.
  - The wires are immersed in a liquid-filled well whose temperature is uniform at $T_{hot}$ over a long distance to prevent conduction heat transfer along the wires, which would cause the sensing tip to read too low.
- When used to measure the temperature of a hot gas, the wires are in contact with a cool duct wall, conduction is now not negligible, and the sensing tip will read low.
Calibration Accuracy versus Installed Accuracy

• If such an error causes unacceptable uncertainty, we may measure the wall temperature, estimate the needed heat transfer parameter, and compute a correction.

• This correction will improve the uncertainty, but not eliminate it, since the correction will itself be uncertain, which uncertainty we will have to estimate and include.

• For example, if our reading is 357°C and,
  ▫ the correction is +8°C, the nominal value is 365°C.
  ▫ If the uncertainty in the correction is ±2°C, and the uncertainty due to other sources was ±5°C, the temperature would be quoted as 365±7°C.
Calibration Accuracy versus Installed Accuracy

- **In situ calibration**, if possible, would in many cases be preferred,
  - the calibration numerical results would include all the effects contributing to uncertainty
  - not require separate judgments and estimates based on experience rather than actual measured data.
- **In a similar spirit, we should also consider the end-to-end calibration.**
  - rather than calibrating separately each link (sensor, amplifier, filter, recorder, etc.) in our measurement chain and then combining the individual uncertainties mathematically, we apply a standard to only the sensor input and record only the final output.
  - **Advantage**: all interactions among the links are automatically taken into account and the procedure may be considerably quicker.
  - **Disadvantage**: we do not see which components are contributing the most to the total uncertainty.
  - Even when we do perform the individual calibrations, a final end-to-end study may be desirable.
Combination of Component Errors in Overall System-Accuracy Calculations

- A measurement system is often made up of a chain of components, each of which is subject to individual (known) inaccuracy.

\[
y = f(x_1, x_2, x_3, \ldots x_n)
\]

\[
\Delta y \approx \frac{df}{dx_1} \cdot \Delta x_1 + \frac{df}{dx_2} \cdot \Delta x_2 + \frac{df}{dx_3} \cdot \Delta x_3 + \ldots + \frac{df}{dx_n} \cdot \Delta x_n
\]

- How is the overall inaccuracy computed?

- If the \( \Delta x \)'s are now considered to be the uncertainties \( u_{x_i} \) in each measured value \( x_i \), then the corresponding uncertainty \( U_y \) in \( y \) is

\[
U_y \approx \sqrt{\left(\frac{df}{dx_1} \cdot u_{x_1}\right)^2 + \left(\frac{df}{dx_2} \cdot u_{x_2}\right)^2 + \left(\frac{df}{dx_3} \cdot u_{x_3}\right)^2 + \ldots + \left(\frac{df}{dx_n} \cdot u_{x_n}\right)^2}
\]
Combination of Component Errors in Overall System-Accuracy Calculations

- Think of the partial derivative as the sensitivity of $y$ to changes in the particular $x$.

$$\Delta y \approx \frac{\partial f}{\partial x_1} \cdot \Delta x_1 + \frac{\partial f}{\partial x_2} \cdot \Delta x_2 + \frac{\partial f}{\partial x_3} \cdot \Delta x_3 + \cdots + \frac{\partial f}{\partial x_n} \cdot \Delta x_n$$

- when a partial derivative has a large numerical value, $y$ is very sensitive to that particular $x$.
- The partial derivatives are numerically evaluated at the operating point,
  - they are constants (not functions), so this equation defines $y$ as a linear function of the $x$'s, even though the original function ($f$) may be nonlinear.
Combination of Component Errors in Overall System-Accuracy Calculations

- This relation is called the *root-sum-square (rss)* formula,

\[ U_y \approx \sqrt{\left(\frac{df}{dx_1} \cdot u_{x_1}\right)^2 + \left(\frac{df}{dx_2} \cdot u_{x_2}\right)^2 + \left(\frac{df}{dx_3} \cdot u_{x_3}\right)^2 + \cdots + \left(\frac{df}{dx_n} \cdot u_{x_n}\right)^2} \]

- We have not here proven its validity, but this rests on the fact that the standard deviation of any linear function of Gaussian independent variables is given by the square root of the sum of squares of the individual standard deviations.
- It is an approximate result because \( y \) is not really a linear function of the \( x \)'s; it is close to linear only for small changes.
Tuning Manometers Uncertainty

- Manometers pressure-measuring devices determine a pressure by measuring the height of a column of fluid \( (P=\rho gh) \). We would like to achieve an accuracy of 0.1% of the maximum reading, 10 kPa. This is to be done by using a type of manometer called a well manometer, which has an uncertainty of 1/10 mm in reading the scale.

Estimate the uncertainty that can be tolerated in the density of a gage fluid, which has a nominal value of 2500 kg/m\(^3\). It is assumed that the value of \( g \) is known to a much higher degree of accuracy than the rest of the parameters.

\[
\delta P = \pm \left( \frac{\partial P}{\partial \rho} \delta \rho \right) \pm \left( \frac{\partial P}{\partial g} \delta g \right) \pm \left( \frac{\partial P}{\partial h} \delta h \right),
\]

\[
\delta P = \pm (h g \delta \rho) \pm (h \rho \delta g) \pm (g \rho \delta h),
\]

\[
\frac{\delta P}{P} = \pm \left( \frac{\delta \rho}{\rho} \right) \pm \left( \frac{\delta g}{g} \right) \pm \left( \frac{\delta h}{h} \right).
\]

\[
h = \frac{P}{\rho g} = \frac{10000}{2500 \times 9.81} = 0.408m \quad \Rightarrow
\]

\[
\left( \frac{1}{1000} \right)^2 = \left( \frac{\delta \rho}{\rho} \right)^2 + \left( \frac{0.8}{9.81} \right)^2 + \left( \frac{1}{4080} \right)^2
\]

\[
\frac{\delta \rho}{\rho} = 0.00097 \Rightarrow 0.1%
\]
Tuning Manometers Uncertainty

• Calculus of variations:

\[ \ln P = \ln \rho + \ln g + \ln h, \]

\[ \frac{\delta P}{P} = \pm \left( \frac{\delta \rho}{\rho} \right) \pm \left( \frac{\delta g}{g} \right) \pm \left( \frac{\delta h}{h} \right), \]

\[ \left( \frac{\delta P}{P} \right)^2 = \left( \frac{\delta \rho}{\rho} \right)^2 + \left( \frac{\delta g}{g} \right)^2 + \left( \frac{\delta h}{h} \right)^2. \]
EXAMPLE: Uncertainty Calculation

- Consider a load transducer made by attaching a strain gage at the root of a cantilever beam.
- A tip force, $P$, will produce a bending strain $\varepsilon_x$.
- Assume the electrical output of the strain gage is $e = K\varepsilon_x e_o$ where $e_o$ is the excitation voltage and $K$ is the calibration factor to be determined.
Uncertainty Calculation

\[
K = \frac{e}{\varepsilon e_0} = \frac{ebh^2E}{e_06PL}
\]

\[
\delta K = \left[ \left( \frac{\partial K}{\partial e} \delta e \right)^2 + \left( \frac{\partial K}{\partial e_0} \delta e_0 \right)^2 + \left( \frac{\partial K}{\partial b} \delta b \right)^2 + \left( \frac{\partial K}{\partial h} \delta h \right)^2 + \left( \frac{\partial K}{\partial E} \delta E \right)^2 + \left( \frac{\partial K}{\partial P} \delta P \right)^2 + \left( \frac{\partial K}{\partial L} \delta L \right)^2 \right]^{1/2}
\]

\[
\frac{\delta K}{K} = \left[ \left( \frac{\delta e}{e} \right)^2 + \left( \frac{\delta e_0}{e_0} \right)^2 + \left( \frac{\delta b}{b} \right)^2 + \left( 2 \frac{\delta h}{h} \right)^2 + \left( \frac{\delta E}{E} \right)^2 + \left( \frac{\delta P}{P} \right)^2 + \left( \frac{\delta L}{L} \right)^2 \right]^{1/2}
\]
Uncertainty Calculation

- Assume the following uncertainties:

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<tr>
<th>Variable</th>
<th>Bias ($u_x/x$)</th>
<th>Precision ($u_x/x$)</th>
<th>Notes</th>
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<td>Digital DVM yields good precision</td>
</tr>
<tr>
<td>$e_0$</td>
<td>0.5%</td>
<td>0</td>
<td>Only single initial measurement made</td>
</tr>
<tr>
<td>$b$</td>
<td>0.5%</td>
<td>0</td>
<td>same</td>
</tr>
<tr>
<td>$h$</td>
<td>0.5%</td>
<td>0</td>
<td>same</td>
</tr>
<tr>
<td>$E$</td>
<td>2%</td>
<td>0</td>
<td>same</td>
</tr>
<tr>
<td>$P$</td>
<td>1%</td>
<td>2%</td>
<td>Bias reflects calibration errors while Precision is random error in readings</td>
</tr>
<tr>
<td>$L$</td>
<td>0.2</td>
<td>0</td>
<td>Only single initial measurement made</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>2.65%</td>
<td>2.01%</td>
<td></td>
</tr>
<tr>
<td>Single sample</td>
<td>2.65%</td>
<td>3.94%</td>
<td></td>
</tr>
</tbody>
</table>

Note: for 95% confidence interval (1.96 $\sigma$); this affects only the precision error which is purely random.
Uncertainty Calculation

- Total uncertainty in the measurement of \( K \) is then:

\[
\frac{\delta K}{K} = \left[ U_{bias}^2 + U_{precision}^2 \right]^{1/2}
\]

\[= \left[ 2.65^2 + 3.94^2 \right]^{1/2} \% = 4.75\% \]