Parameter retrieval of chiral metamaterials based on the state-space approach

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This paper deals with the introduction of an approach for the electromagnetic characterization of homogeneous chiral layers. The proposed method is based on the state-space approach and properties of a 4×4 state transition matrix. Based on this, first, the forward problem analysis through the state-space method is reviewed and properties of the state transition matrix of a chiral layer are presented and proved as two theorems. The formulation of a proposed electromagnetic characterization method is then presented. In this method, scattering data for a linearly polarized plane wave incident normally on a homogeneous chiral slab are combined with properties of a state transition matrix and provide a powerful characterization method. The main difference with respect to other well-established retrieval procedures based on the use of the scattering parameters relies on the direct computation of the transfer matrix of the slab as opposed to the conventional calculation of the propagation constant and impedance of the modes supported by the medium. The proposed approach allows avoiding nonlinearity of the problem but requires getting enough equations to fulfill the task which was provided by considering some properties of the state transition matrix. To demonstrate the applicability and validity of the method, the constitutive parameters of two well-known dispersive chiral metamaterial structures at microwave frequencies are retrieved. The results show that the proposed method is robust and reliable.

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I. INTRODUCTION

The interaction of electromagnetic waves with chiral media has long been a subject of interest. Chiral media have two important properties: namely optical activity which may rotate the polarization plane of a linearly polarized wave propagating through it, and circular dichroism which is attributed to the different absorptivity of right- and left-circularly polarized waves inside the chiral medium [1]. Recently, more and more interest has been focused on the chiral metamaterials (CMMs) due to their attractive properties such as strong optical activity and negative refractive index in microwave and optics. CMMs are metamaterials (MTMs) made of unit cells without any mirror symmetry. This property results in the breaking degeneracy of two circularly polarized waves, i.e., right- and left-circularly polarized waves with different refractive indices. Negative refraction can be realized in CMMs, with neither negative permittivity nor permeability required. Based on this concept, several semiplanar structures with fourfold rotational symmetry have been suggested as the different kinds of CMMs [2–13]. In addition, numerous applications have been proposed for chiral metamaterials such as realizing structures with strong optical activity [14–16], realizing negative refraction index in the terahertz regime [17,18], circular polarizer [19,20], wide angle and polarization independent microwave absorber [21], and gain enhancement and axial ratio improvement of circularly polarized antennas [22].

Retrieving the effective medium parameters of MTMs is a critical concept in MTMs design and can be achieved by several approaches of which the S-parameters method is the most prevalent one [23–25]. This approach has been modified in [26] in order to extract effective parameters of CMMs by considering a CMM slab with thickness d illuminated by normal incidence of circularly polarized waves. Recently, an improved algorithm was derived based on Kramers-Kronig relations for the unique extraction of effective parameters of CMMs [27]. Briefly, in the standard full-wave parameter retrieval methods, the eigenwaves of the wave equation in an isotropic chiral layer are derived and the general form of electric and magnetic fields inside such an environment are determined. Then, using the boundary conditions of the problem, the constitutive parameters of the CMM structure at normal incidence are retrieved.

The objective of the present work is to present an alternative parameter retrieval method for the electromagnetic characterization of chiral slabs using the state transition matrix method. The transition matrix method is commonly used to deal with the problems of plane wave scattering from planar layered generalized anisotropic or bianisotropic media, and its application in forward scattering problems has been well studied over the years [28–33]. In this work we apply the formulation to inverse scattering problems. The proposed retrieval method which uses scattering data corresponding to linearly polarized waves is mainly based on the properties of the state transition matrix of an isotropic chiral layer which are discussed as two theorems.

The paper is organized as follows. Section II is dedicated to review of the forward problem analysis using the state transition matrix method. Two properties of the state transition matrix of a chiral layer are presented in Sec. III. Section IV deals with the detailed formulation of parameter retrieval

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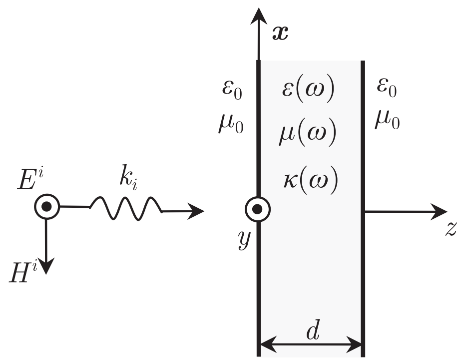

FIG. 1. A homogeneous chiral slab exposed to a linearly polarized plane wave.

technique and its accuracy is verified along some example computations in Sec. V. Finally, a summary and conclusions are provided in Sec. VI.

II. FORWARD PROBLEM ANALYSIS USING THE STATE-SPACE METHOD

Assuming a time harmonic field with $e^{j\omega t}$ dependence, isotropic chiral media are characterized by the following constitutive relations [1]:

$$\begin{pmatrix} \bar{D} \\ \bar{B} \end{pmatrix} = \begin{pmatrix} \varepsilon_0\varepsilon_r & -j\kappa/c \\ j\kappa/c & \mu_0\mu_r \end{pmatrix} \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix}, \tag{1}$$

where ε_0 and μ_0 are the permittivity and permeability of vacuum, c is the speed of light in free space, ε_r and μ_r are the relative permittivity and relative permeability of the chiral medium, respectively, and κ is the so-called chirality parameter which rotates electric and magnetic fields. It is assumed that a linearly polarized wave is incident normally from free space to the chiral slab as shown in Fig. 1. The planar structure is of infinite extent along the y direction, and so the derivative of the fields with respect to the y variable vanishes. In addition, the derivative of the fields with respect to the x variable in the slab must take on the same value as in free space in order to satisfy the boundary conditions on tangential fields at the boundaries. So, due to the normal incidence of a plane wave, $\partial/\partial x = 0$. The most important feature of the state-space approach is that no matter how complex the medium under study is, the transverse components of electric and magnetic fields with some algebraic manipulating become four coupled, first-order differential equations. Here, by substituting the constitutive equations of chiral media into curl Maxwell's equations and by eliminating z components of electric and magnetic fields one can write

$$\frac{d}{dz}\begin{pmatrix} \bar{E}_T \\ \bar{H}_T \end{pmatrix} = \Gamma_\omega \begin{pmatrix} \bar{E}_T \\ \bar{H}_T \end{pmatrix}, \tag{2}$$

where $\bar{E}_T = (E_x, E_y)$ and $\bar{H}_T = (H_x, H_y)$ are the transverse components of electric and magnetic fields, respectively, and the elements of the 4×4 Γ_ω matrix are given by

$$\Gamma_\omega = \frac{\omega}{c}\Gamma = \begin{pmatrix} 0 & \frac{\omega}{c}\kappa & 0 & -j\omega\,\mu_0\mu_r \\ -\frac{\omega}{c}\kappa & 0 & j\omega\,\mu_0\mu_r & 0 \\ 0 & j\omega\,\varepsilon_0\varepsilon_r & 0 & \frac{\omega}{c}\kappa \\ -j\omega\,\varepsilon_0\varepsilon_r & 0 & -\frac{\omega}{c}\kappa & 0 \end{pmatrix}, \tag{3}$$

where ω is the angular frequency.

Defining a 4×4 state transition matrix Φ that relates the transverse components of electric and magnetic fields at the two boundaries of the chiral slab,

$$\begin{pmatrix} \bar{E}_T(0) \\ \bar{H}_T(0) \end{pmatrix} = \Phi \begin{pmatrix} \bar{E}_T(d) \\ \bar{H}_T(d) \end{pmatrix} = \begin{pmatrix} (\Phi_1)_{2\times2} & (\Phi_2)_{2\times2} \\ (\Phi_3)_{2\times2} & (\Phi_4)_{2\times2} \end{pmatrix} \begin{pmatrix} \bar{E}_T(d) \\ \bar{H}_T(d) \end{pmatrix}, \tag{4}$$

where the state transition matrix Φ is given by

$$\Phi = e^{-\Gamma_\omega d}. \tag{5}$$

For the computation of the exponential of a square matrix, many methods have been proposed [34], such as expansion of Φ in a power series and the Cayley-Hamilton theorem.

By introducing the reflection and transmission matrices, R and T, we can write

$$\bar{E}_T^r(z=0) = \begin{pmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{pmatrix} \bar{E}_T^i(z=0), \tag{6}$$

$$\bar{E}_T^t(z=d) = \begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix} \bar{E}_T^i(z=0), \tag{7}$$

where the superscripts i, r, and t denote the incident, reflected, and transmitted field, respectively. Using (4), one can write [28]

$$\bar{E}_T^i(0) + \bar{E}_T^r(0) = \Phi_1\bar{E}_T^t(d) + \Phi_2 Z_0^{-1}\bar{E}_T^t(d), \tag{8}$$

$$Z_0^{-1}\bar{E}_T^i(0) - Z_0^{-1}\bar{E}_T^r(0) = \Phi_3\bar{E}_T^t(d) + \Phi_4 Z_0^{-1}\bar{E}_T^t(d), \tag{9}$$

where wave impedance matrix Z_0 is defined as

$$Z_0 = \begin{pmatrix} 0 & \eta_0 \\ -\eta_0 & 0 \end{pmatrix}, \tag{10}$$

where η_0 is the intrinsic impedance of free space. Using (8) and (9) and by considering (6) and (7), after some matrix manipulations one can write [28]

$$R = [\Phi_1 Z_0 + \Phi_2 - Z_0(\Phi_3 Z_0 + \Phi_4)] \times [\Phi_1 Z_0 + \Phi_2 + Z_0(\Phi_3 Z_0 + \Phi_4)]^{-1}, \tag{11}$$

$$T = 2Z_0[\Phi_1 Z_0 + \Phi_2 + Z_0(\Phi_3 Z_0 + \Phi_4)]^{-1}. \tag{12}$$

III. PROPERTIES OF THE STATE TRANSITION MATRIX OF AN ISOTROPIC CHIRAL LAYER

The state transition matrix Φ of an isotropic chiral slab has various properties, some of which are useful for the parameter retrieval algorithm. The following two useful theorems are introduced here and then proved in the Appendixes.

Theorem 1. Determinant of the state transition matrix of an isotropic chiral slab is equal to unity. Proof of this theorem is presented in Appendix A.

Theorem 2. State transition matrix of an isotropic chiral slab could be written as

$$\Phi = \begin{pmatrix} \boxed{\Phi_{11}} & \boxed{\Phi_{12}} & \boxed{\Phi_{13}} & \boxed{\Phi_{14}} \\ -\Phi_{12} & \Phi_{11} & -\Phi_{14} & \Phi_{13} \\ \boxed{\Phi_{31}} & \boxed{\Phi_{32}} & \Phi_{11} & \Phi_{12} \\ -\Phi_{32} & \Phi_{31} & -\Phi_{12} & \Phi_{11} \end{pmatrix}, \tag{13}$$

which has only six distinct elements: Φ_{11}, Φ_{12}, Φ_{13}, Φ_{14}, Φ_{31}, and Φ_{32}, given in Appendix B. In addition, the absolute of any element of Φ^{-1} compared with that of the state transition matrix Φ remains unchanged, and as proved in Appendix B, the Φ^{-1} matrix could be written as the following:

$$\Phi = \begin{pmatrix} \boxed{\Phi_{11}} & -\boxed{\Phi_{12}} & \boxed{\Phi_{13}} & -\boxed{\Phi_{14}} \\ \Phi_{12} & \Phi_{11} & \Phi_{14} & \Phi_{13} \\ \boxed{\Phi_{31}} & -\boxed{\Phi_{32}} & \Phi_{11} & -\Phi_{12} \\ \Phi_{32} & \Phi_{31} & \Phi_{12} & \Phi_{11} \end{pmatrix}. \tag{14}$$

IV. PARAMETER RETRIEVAL TECHNIQUE FORMULATION

In this section, we illustrate the proposed procedure for retrieving the constitutive parameters of CMMs using scattering parameters, based on the state transition matrix and its properties.

It is clear that all of the constitutive parameters of a chiral slab are active when the slab is normally illuminated by linearly x- or y-polarized plane waves, and so, for retrieving all the constitutive parameters of a chiral slab, scattering parameters at normal incidence are required. Assuming the incident wave as the form of $\bar{E}_i = e^{-jk_0 z}\hat{a}_y$ where $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the free space wave number, (8) and (9) may be rewritten as

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} E_x^r \\ E_y^r \end{pmatrix} = \left[\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ -\Phi_{12} & \Phi_{11} \end{pmatrix} + \frac{1}{\eta_0}\begin{pmatrix} \Phi_{14} & -\Phi_{13} \\ \Phi_{13} & \Phi_{14} \end{pmatrix}\right] \times \begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}, \tag{15}$$

$$\begin{pmatrix} -1/\eta_0 \\ 0 \end{pmatrix} - \frac{1}{\eta_0}\begin{pmatrix} -E_y^r \\ E_x^r \end{pmatrix} = \left[\begin{pmatrix} \Phi_{31} & \Phi_{32} \\ -\Phi_{32} & \Phi_{31} \end{pmatrix} + \frac{1}{\eta_0}\begin{pmatrix} \Phi_{12} & -\Phi_{11} \\ \Phi_{11} & \Phi_{12} \end{pmatrix}\right]\begin{pmatrix} E_x^t \\ E_y^t \end{pmatrix}. \tag{16}$$

Therefore, the following set of four linear equations is derived:

$$\begin{aligned} E_x^r &= (\Phi_{11}+\Phi_{14}/\eta_0)E_x^t + (\Phi_{12} - \Phi_{13}/\eta_0)E_y^t \\ 1 + E_y^r &= (-\Phi_{12} + \Phi_{13}/\eta_0)E_x^t + (\Phi_{11} + \Phi_{14}/\eta_0)E_y^t \\ 1 - E_y^r &= -(\eta_0\Phi_{31} + \Phi_{12})\,E_x^t - (\eta_0\Phi_{32} - \Phi_{11})\,E_y^t \\ E_x^r &= (\eta_0\Phi_{32} - \Phi_{11})\,E_x^t - (\eta_0\Phi_{31} + \Phi_{12})\,E_y^t. \end{aligned} \tag{17}$$

It should be noticed that, when a chiral slab is illuminated normally by a linearly polarized plane wave, the polarization of the reflected wave is the same as that of the incident wave, that is, the chiral slab behaves as an ordinary dielectric as far as the reflected wave is concerned [1]. Therefore $E_x^r = 0$. It is known that the goal of the first step in the inverse procedure is to compute the state transition matrix Φ. Observe that using the reflection and transmission coefficients, a simple set of four linear equations has been derived, while the number of distinct elements of the state transition matrix Φ is 6. It seems that due to this unbalance, application of the state transition matrix method in such inverse problems has remained an almost untouched topic in the literature. The proposed method for balancing the number of unknowns and equations is to use properties of the state transition matrix of an isotropic chiral slab discussed in the prior section. In fact, by considering the presented theorems, the necessary and sufficient equations are provided to determine uniquely the unknown elements of matrix Φ.

Once the state transition matrix Φ is determined, one can write

$$\Gamma = -\frac{c}{\omega d}\ln(\Phi) = -\frac{\lambda_0}{2\pi d}\ln(\Phi), \tag{18}$$

where λ_0 is the free space wavelength. For the computation of the logarithm of a square matrix, we can use the Cayley-Hamilton theorem discussed in Appendix B.

It is interesting to investigate the source of ambiguity and branch selecting problem in the presented parameter retrieval technique. Clearly, for the Γ matrix with eigenvalues γ_1, γ_2, γ_3, and γ_4, and eigenvectors $\bar{\mathrm{V}}_1$, $\bar{\mathrm{V}}_2$, $\bar{\mathrm{V}}_3$, and $\bar{\mathrm{V}}_4$, there exists a matrix M such that $\Gamma = \mathrm{M}\Lambda\mathrm{M}^{-1}$ where Λ is a diagonal matrix as the form of $\mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ and

$$\mathrm{M} = [\bar{\mathrm{V}}_1, \bar{\mathrm{V}}_2, \bar{\mathrm{V}}_3, \bar{\mathrm{V}}_4]. \tag{19}$$

As stated in (A3), we can write

$$\Phi = \mathrm{M}\,\mathrm{diag}(e^{-\gamma_1\omega d/c}, e^{-\gamma_2\omega d/c}, e^{-\gamma_3\omega d/c}, e^{-\gamma_4\omega d/c})\mathrm{M}^{-1}, \tag{20}$$

and so Γ and Φ have the same set of eigenvectors $\bar{\mathrm{V}}_\mathrm{n}$. In the inverse problem, after determining the state transition matrix Φ, its eigenvalues λ_n ($n = 1$, 2, 3, and 4) and eigenvectors are fully determined. Notice that for the computation of the Γ matrix through $\mathrm{M}\Lambda\mathrm{M}^{-1}$, the eigenvalues γ_n should be determined by

$$\gamma_n = -\frac{c}{\omega d}\ln\lambda_n = -\frac{c}{\omega d}[\ln|\lambda_n| + j\,(\arg\lambda_n + 2m\pi)], \tag{21}$$

in which m is an arbitrary integer number at any frequency. The resulting uncertainty due to the existence of m in (21) is referred to as a branching problem, which is due to the multibranch form of the complex logarithmic function. If the thickness d is much smaller than the wavelength, the eigenvalues γ_n and consequently the Γ matrix are unambiguously identified.

Finally, the characterization of the medium is then performed by matching the Γ matrix obtained from the scattering coefficients to its expression in terms of the constitutive parameters presented in (3).

Notice that the main difference with respect to other well-established retrieval procedures based on the use of the scattering parameters relies on the direct computation of the transfer matrix of the slab as opposed to the conventional calculation of the propagation constant and impedance of the modes supported by the medium. In conventional retrieval procedures [26], one should directly solve a set of nonlinear equations to obtain closed formulas for the constitutive electromagnetic parameters of the chiral slab in terms of scattering data corresponding to circularly polarized plane waves. It seems that this subject is the weak point of conventional retrieval procedures that is even much worse in more complex media. In fact, the retrieval procedures using scattering data do not yield a unique result, and so we need the aforesaid closed formulas for the exact identification of factors causing uncertainty and ambiguity in the results. This kind of ambiguity in CMMs arises from the existence of a multibranch logarithmic function in the relations [26]. In the proposed method based on the state-space approach, it is not necessary to obtain analytical formulas for unknown electromagnetic parameters in terms of scattering parameters and only some simple matrix calculations should be done. To sum up, the proposed retrieval procedure in the present manuscript allows avoiding nonlinearity of the problem but requires getting enough equations to fulfill the task which was provided by considering some properties of the state transition matrix. Although these different approaches are generally equivalent, the proposed procedure may be convenient from computational and practical points of view.

V. NUMERICAL EXAMPLES AND RESULTS

In order to illustrate the applicability of the proposed method for the reconstruction of the constitutive parameters of chiral layers, two examples are provided here. In these numerical examples, we use the Padé approximation with scaling and squaring to save computational time in handling the exponential and logarithms of the matrices.

A. Dispersive chiral slab

Consider a dispersive chiral layer with thickness $d = 1.6$ mm (Fig. 1) whose constitutive parameters have the following forms of frequency dependence (assuming $e^{j\omega t}$ time dependence) [27]:

$$\varepsilon = \varepsilon_b - \frac{\Omega_{\varepsilon 1} f_{10}^2}{f^2 - f_{10}^2 - jf\, f_{10}\,\Gamma_1} - \frac{\Omega_{\varepsilon 2} f_{20}^2}{f^2 - f_{20}^2 - jf\, f_{20}\,\Gamma_2}, \tag{22}$$

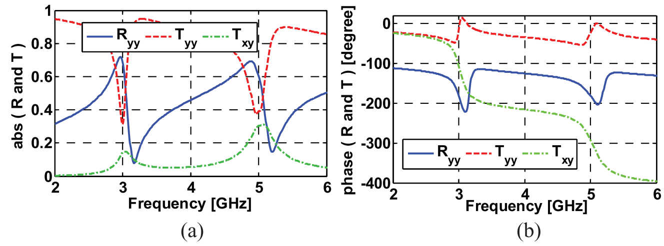

FIG. 2. (Color online) (a) Amplitude and (b) phase of reflection and transmission coefficients of dispersive chiral slab with thickness $d = 1.6$ mm.

$$\mu = \mu_b - \frac{\Omega_{\mu 1} f^2}{f^2 - f_{10}^2 - jf\, f_{10}\,\Gamma_1} - \frac{\Omega_{\mu 2} f^2}{f^2 - f_{20}^2 - jf\, f_{20}\,\Gamma_2}, \tag{23}$$

$$\kappa = -\frac{\Omega_{\kappa 1} f_{10}\, f}{f^2 - f_{10}^2 - jf\, f_{10}\,\Gamma_1} + \frac{\Omega_{\kappa 2} f_{20}\, f}{f^2 - f_{20}^2 - jf\, f_{20}\,\Gamma_2}, \tag{24}$$

where $f_{10} = 3$ GHz, $f_{20} = 5$ GHz, $\varepsilon_b = 8.52$, and $\mu_b = 0.89$ are resonant frequencies and background relative permittivity and permeability, respectively. Other parameters, $\Gamma_1 = 0.020$, $\Gamma_2 = 0.022$, $\Omega_{\varepsilon 1} = 0.96$, $\Omega_{\varepsilon 2} = 0.58$, $\Omega_{\mu 1} = 0.007$, $\Omega_{\mu 2} = 0.024$, $\Omega_{\kappa 1} = 0.097$, and $\Omega_{\kappa 2} = 0.136$ are coefficients describing strength of resonances.

The transmission and reflection coefficients corresponding to linearly polarized incident waves obtained by the presented formulation in Sec. II are shown in Fig. 2. The proposed reconstruction technique based on the state transition matrix method is applied to these data and the discussed set of equations has been simply solved and the Φ and Γ matrices are determined. Eigenvalues of these matrices are shown in Fig. 3. Notice that with considering (18) and due to the small thickness of slab compared to the free space wavelength, the eigenvalues of the Γ matrix are continuous and no branch selection problem occurred. The retrieved constitutive parameters of the chiral

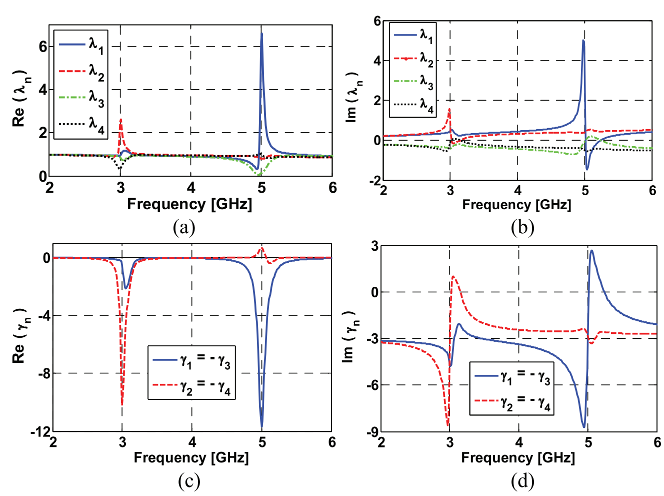

FIG. 3. (Color online) Real and imaginary parts of eigenvalues of (a) and (b) the Φ matrix, and (c) and (d) the Γ matrix for dispersive chiral slab with thickness $d = 1.6$ mm.

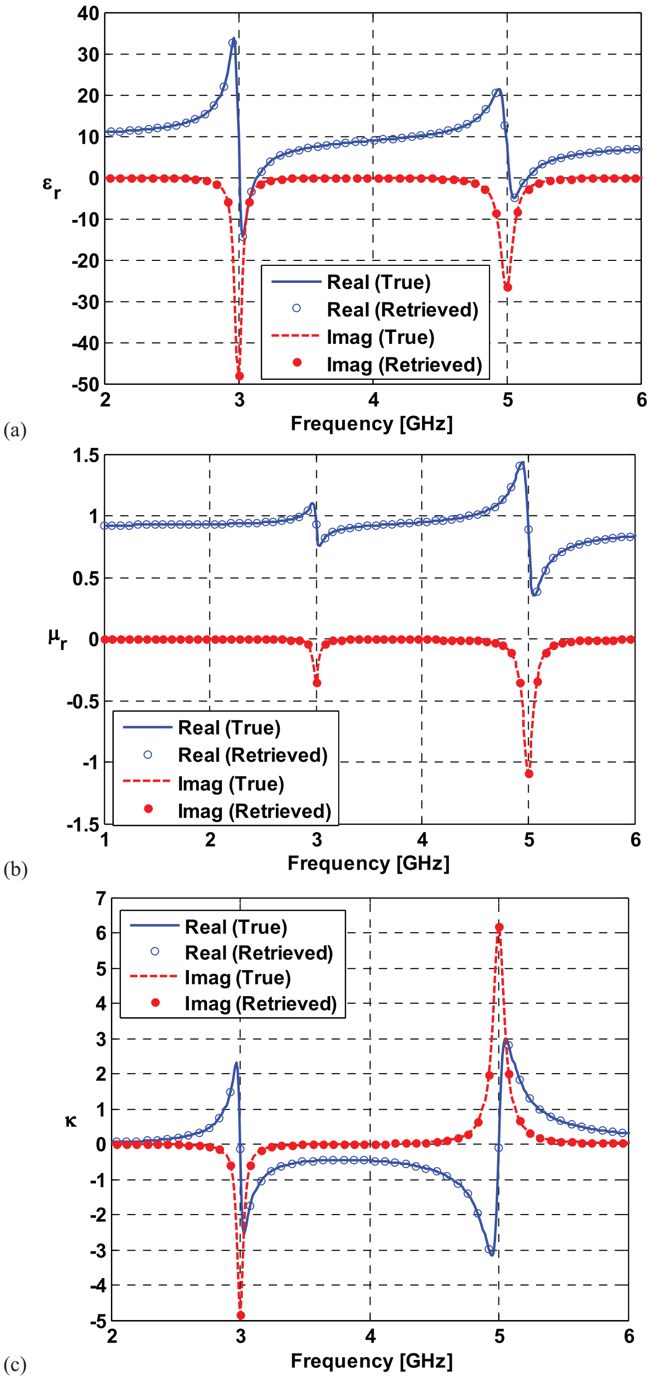

FIG. 4. (Color online) Real and imaginary parts of true and retrieved constitutive tensor parameters of the dispersive chiral slab. (a) Relative permittivity. (b) Relative permeability. (c) Chirality parameter.

slab are compared with true ones in Fig. 4. The comparison between the results illustrates the excellent behavior of the proposed technique.

Now, consider a further study, a 6.4-mm-thick chiral slab which is four times thicker than the former one. The amplitudes and phases of reflection and transmission coefficients at normal incidence are shown in Fig. 5. Imaginary parts of eigenvalues of the Γ matrix are shown in Fig. 6. It is noteworthy that the real parts of these eigenvalues are exactly the same as the previous

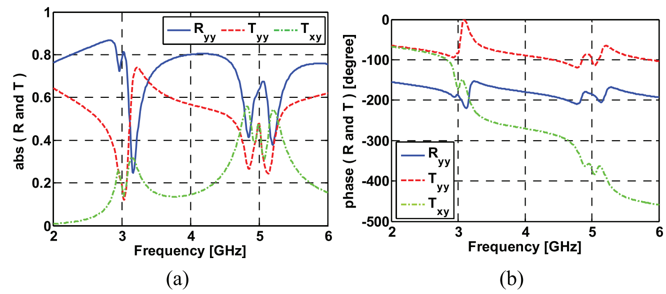

FIG. 5. (Color online) (a) Amplitude and (b) phase of reflection and transmission coefficients of dispersive chiral slab with thickness $d = 6.4$ mm.

values shown in Fig. 3(c). Notice that, unlike the prior case, there are considerable discontinuities in γ_n curves which may be attributed to the branching problem in (21). Considering (3), it can be easily seen that $\gamma_1 = -\gamma_3 = -j(\sqrt{\mu_r \varepsilon_r} + \kappa)$ and $\gamma_2 = -\gamma_4 = -j(\sqrt{\mu_r \varepsilon_r} - \kappa)$, and so one can use the Kramers-Kronig (K-K) relations discussed in [24] to ensure accuracy of the results. However, no discontinuity is seen in the results obtained by K-K relations. Therefore, considering K-K solutions, the discontinuities which have an allowable value $2\pi c/\omega d$ about 3 GHz should be removed and the nearest solution to the K-K one is achieved. The modified curves are also shown in Fig. 6. Observe that as expected, the

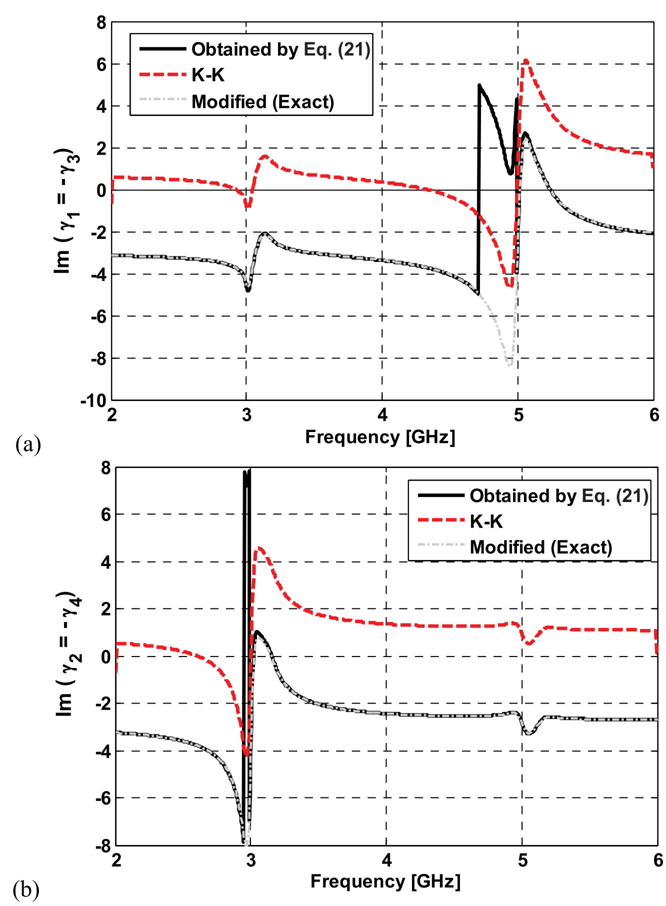

FIG. 6. (Color online) Imaginary parts of eigenvalues of the Γ matrix for dispersive chiral slab with thickness $d = 6.4$ mm.

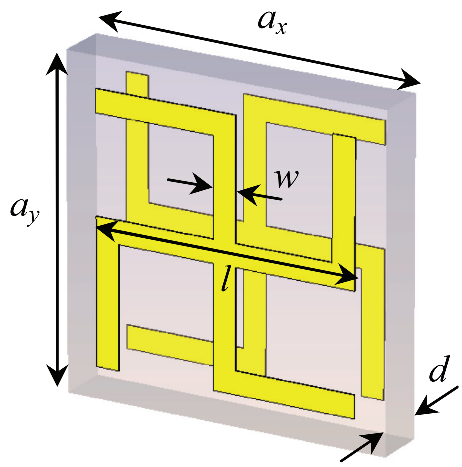

FIG. 7. (Color online) Unit cell of the conjugated gammadions structure [7]. The parameters are given by $a_x = a_y = 10$ mm, $d = 1.6$ mm, $w = 0.7$ mm, and $l = 8.1$ mm.

obtained eigenvalues of the Γ matrix are independent of slab thickness and are the same as the previous values. After unique determination of the Γ matrix, the constitutive parameters of the chiral slab are computed, which are identical to the previous ones as shown in Fig. 4.

B. Conjugated gammadions CMM structure

The layout of a well-known CMM structure is shown in Fig. 7. The unit cell of this structure is constructed of two copper conjugated gammadions patterned on the opposite sides of an FR-4 board with relative dielectric constant 4.2 and dielectric loss tangent 0.02 [7]. This structure exhibits uniaxial chirality for the normal incident electromagnetic waves.

Numerical simulations are performed by using the unit cell template and the frequency domain solver of the CST MICROWAVE STUDIO. The amplitudes and phases of transmission and reflection coefficients are shown in Figs. 8(a) and 8(b). The proposed retrieval method based on the state transition matrix method is applied to these data and Φ and Γ matrices are determined. Real and imaginary parts of eigenvalues of Φ and Γ matrices are shown in Figs. 9 and 10, respectively.

Once the eigenvalues of the Φ matrix are determined, the eigenvalues of the Γ matrix could be identified using (21). Observe that there are two discontinuity points in these

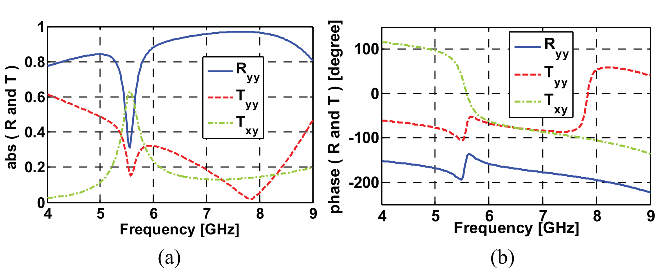

FIG. 8. (Color online) (a) Amplitude and (b) phase of reflection and transmission coefficients of conjugated gammadions CMM structure.

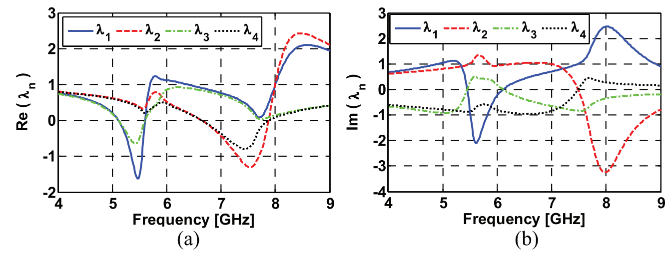

FIG. 9. (Color online) Real and imaginary parts of eigenvalues of the Φ matrix for conjugated gammadions CMM structure.

eigenvalue curves at the resonant frequencies of 5.6 and 7.8 GHz. The approximate solutions obtained by Kramers-Kronig relations are also shown in Fig. 10(b). Finally, with comparing the founded Γ matrix with (3), the relative permittivity, relative permeability, and chirality parameter were determined and are shown in Fig. 11. Observe that there is very good agreement between the obtained results by the state-space approach and previously reported ones in [7] obtained by a standard full-wave method [26].

VI. CONCLUSIONS

An inversion method is proposed to reconstruct the constitutive parameters of a homogeneous chiral slab from the knowledge of co- and cross-reflection and transmission coefficients. The proposed characterization method is based on the state transition matrix and its interesting properties which are discussed as two proven theorems. In order to investigate the applicability of the proposed inversion method, two inversion problems including CMMs are considered and analyzed. The results indicate that the constitutive parameters of CMMs are retrieved successfully. In the presented method, it is not necessary to obtain eigenpolarizations of the chiral layer through solving the wave equation in the isotropic chiral region and then using boundary conditions in order to obtain analytic formulas for the electromagnetic parameters of the chiral slab in terms of scattering parameters. In future, the proposed retrieval method is expected to be used in more complex inversion problems such as electromagnetic characterization of CMMs at oblique incidences.

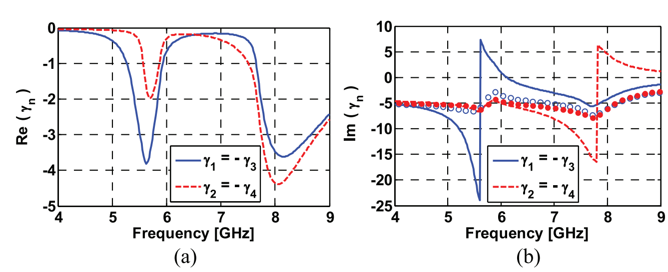

FIG. 10. (Color online) Real and imaginary parts of eigenvalues of the Γ matrix for conjugated gammadions CMM structure. The obtained approximate results based on the Kramers-Kronig relations for $\gamma_1 = -\gamma_3$ and $\gamma_2 = -\gamma_4$ are shown with hollow and solid circles, respectively.

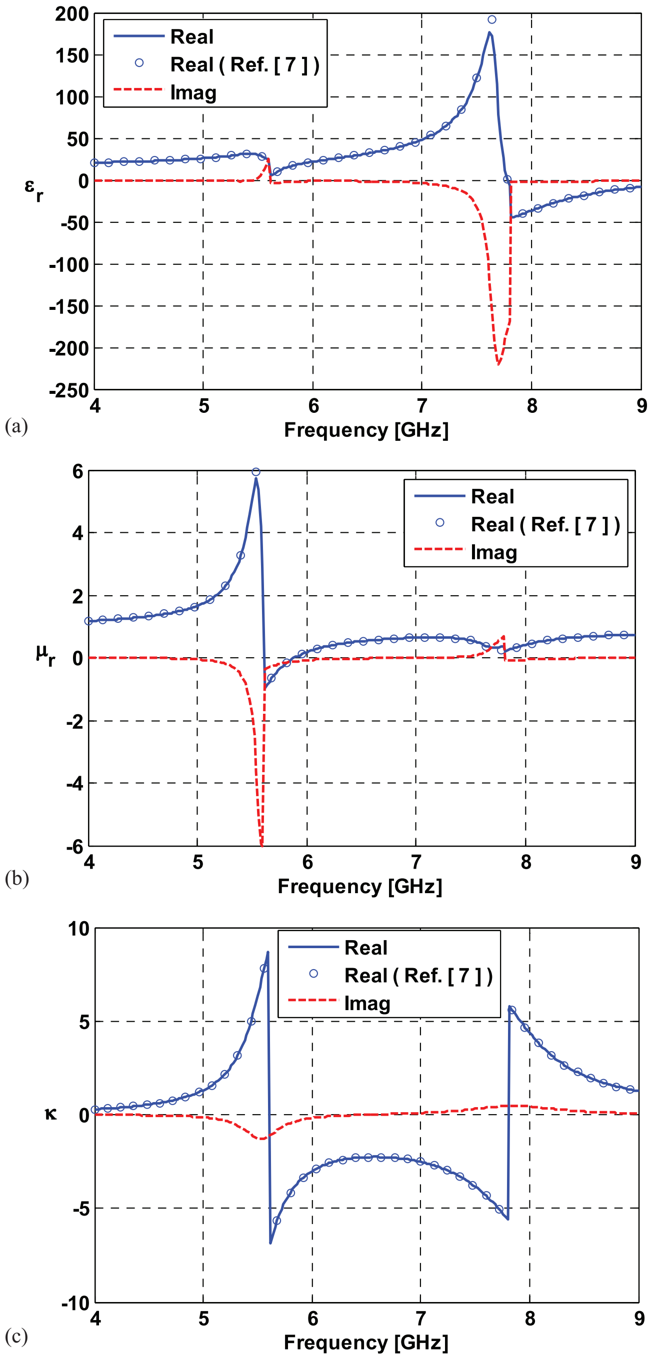

FIG. 11. (Color online) Real and imaginary parts of retrieved constitutive parameters of conjugated gammadions CMM structure. (a) Relative permittivity. (b) Relative permeability. (c) Chirality parameter.

APPENDIX A

For the Γ matrix with eigenvalues γ_1, γ_2, γ_3, and γ_4, there exists a matrix M such that Γ is equal to $\mathrm{M}\Lambda\mathrm{M}^{-1}$ where Λ is a diagonal matrix as the form of diag(γ_1, γ_2, γ_3, γ_4). The exponential function of teh square matrix Γd is defined in terms of an infinite Taylor series form as

$$e^{-\Gamma d} = I - \Gamma d + \frac{1}{2!}\Gamma^2 d^2 - \frac{1}{3!}\Gamma^3 d^3 + \cdots, \qquad \text{(A1)}$$

where I is the 4×4 identity matrix, and the above series converges for all square matrices. By substituting $\mathrm{M}\Lambda\mathrm{M}^{-1}$ into Γ^2, we have

$$\Gamma^2 = (\mathrm{M}\Lambda\mathrm{M}^{-1})(\mathrm{M}\Lambda\mathrm{M}^{-1}) = \mathrm{M}\Lambda(\mathrm{M}^{-1}\mathrm{M})\Lambda\mathrm{M}^{-1} = \mathrm{M}\Lambda^2\mathrm{M}^{-1}, \qquad \text{(A2)}$$

and all powers of Γ are similarly reduced. Thus

$$\begin{aligned} e^{-\Gamma d} &= I - (\mathrm{M}\Lambda\mathrm{M}^{-1})\, d + \frac{1}{2!}(\mathrm{M}\Lambda^2\mathrm{M}^{-1})\, d^2 - \frac{1}{3!}(\mathrm{M}\Lambda^3\mathrm{M}^{-1})\, d^3 + \cdots, \\ &= \mathrm{M}\left[I - \Lambda t + \frac{1}{2!}\Lambda^2 d^2 - \frac{1}{3!}\Lambda^3 d^3 + \cdots\right]\mathrm{M}^{-1} \\ &= \mathrm{M}e^{-\Lambda d}\mathrm{M}^{-1}. \end{aligned} \qquad \text{(A3)}$$

Since the determinant of a product of square matrices is the product of their determinants, the determinant of $e^{-\Gamma d}$ is equal to that of $e^{-\Lambda d}$ which due to the diagonal nature of Λ is $\exp[-(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4)d]$. Beside this, we know that the sum of the eigenvalues of a square matrix is equal to the sum of the diagonal elements of it [34,35]. Thus, according to (6), the determinant of the state transition matrix $\Phi = e^{-\Gamma_\omega d}$ of an isotropic chiral slab is equal to unity.

APPENDIX B

The Cayley-Hamilton theorem relates a matrix to its characteristic polynomial [33]. Using this theorem, the exponential of an arbitrary square matrix can be simply calculated. Assuming a 4×4 W matrix with distinct eigenvalues w_1, w_2, w_3, and w_4, we can write four equations as

$$e^{w_p} = \sum_{q=0}^{3} c_q w_p^q, \qquad \text{(B1)}$$

where $p = 0$, 1, 2, and 3. By solving this set of four linear equations, unknown coefficients c_q for $q = 0$, 1, 2, and 3 are fully determined. Finally, the exponential of W can be written by the following:

$$e^{\mathrm{W}} = c_0\mathrm{I} + c_1\mathrm{W} + c_2\mathrm{W}^2 + c_3\mathrm{W}^3. \qquad \text{(B2)}$$

Assuming $\mathrm{W} = -\Gamma_\omega d, g_1 = j\omega\mu d$, $g_2 = j\omega\varepsilon d$, and $g_3 = \omega\kappa d/c$, the eigenvalues of W are given by

$$w_{1/3} = -w_{2/4} = \left[g_1 g_2 - g_3^2 \pm 2g_3\sqrt{-g_1 g_2}\right]^{1/2}. \qquad \text{(B3)}$$

By substituting these eigenvalues in (B1) and solving the equations, unknown coefficients c_q for $q = 0$, 1, 2, and 3 are fully determined:

$$c_0 = \frac{w_3^2\cosh(w_1) - w_1^2\cosh(w_3)}{w_3^2 - w_1^2}, \qquad \text{(B4)}$$

$$c_1 = \frac{w_3^3\sinh(w_1) - w_1^3\sinh(w_3)}{w_1 w_3\left(w_3^2 - w_1^2\right)}, \qquad \text{(B5)}$$

$$c_2 = -\frac{\cosh(w_1) - \cosh(w_3)}{w_3^2 - w_1^2}, \qquad \text{(B6)}$$

$$c_3 = -\frac{w_3\sinh(w_1) - w_1\sinh(w_3)}{w_1 w_3\left(w_3^2 - w_1^2\right)}. \tag{B7}$$

Then, after some matrix manipulations in (B2), the matrix exponential $\Phi = e^{-\Gamma_\omega d}$ may be written as (21) where

$$\begin{aligned}
\Phi_{11} &= c_0 + c_2\left(g_1 g_2 - g_3^2\right)\\
\Phi_{12} &= -c_1 g_3 - c_3 g_3\left(3g_1 g_2 - g_3^2\right)\\
\Phi_{13} &= 2c_2 g_1 g_3\\
\Phi_{14} &= c_1 g_1 - c_3 g_1\left(3g_3^2 - g_1 g_2\right)\\
\Phi_{31} &= -2c_2 g_2 g_3\\
\Phi_{32} &= -c_1 g_2 + c_3 g_2\left(3g_3^2 - g_1 g_2\right).
\end{aligned} \tag{B8}$$

For the computation of the inverse matrix Φ^{-1}, we can use an identity as $(e^{-\Gamma_\omega d})^{-1} = (e^{\Gamma_\omega d})$, and so compute $e^{\Gamma_\omega d}$. It is evident that the eigenvalues remain unchanged rather than as in the prior case. In this case we should consider

$$e^{-\mathrm{W}} = c_0'\mathrm{I} + c_1'\mathrm{W} + c_2'\mathrm{W}^2 + c_3'\mathrm{W}^3, \tag{B9}$$

and solve four equations,

$$e^{-w_p} = \sum_{q=0}^{3} c_q' w_p^q, \tag{B10}$$

where $p = 0$, 1, 2, and 3. It can be easily seen that the coefficients of c_q' for $q = 0$, 1, 2, and 3 are given by

$$c_0' = c_0,\ \ c_1' = -c_1,\ \ c_2' = c_2,\ \ c_3' = -c_3. \tag{B11}$$

After some matrix manipulations, one can obtain the Φ^{-1} matrix as

$$\Phi^{-1} = \begin{pmatrix} \Phi_{11} & -\Phi_{12} & \Phi_{13} & -\Phi_{14}\\ \Phi_{12} & \Phi_{11} & \Phi_{14} & \Phi_{13}\\ \Phi_{31} & -\Phi_{32} & \Phi_{11} & -\Phi_{12}\\ \Phi_{32} & \Phi_{31} & \Phi_{12} & \Phi_{11} \end{pmatrix}. \tag{B12}$$

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