A Novel Approach for Calculations of Helical Toroidal Coil Inductance Usable in Reactor Plasmas

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Abstract—In this paper, formulas are proposed for the self- and mutual-inductance calculations of a helical toroidal coil by direct and indirect methods at superconductivity conditions. The direct method is based on the Neumann’s equation, and the indirect approach is based on the toroidal and the poloidal components of the magnetic flux density. Numerical calculations show that the direct method is more accurate than the indirect approach at the expense of its longer computational time. Implementation of some engineering assumptions in the indirect method is shown to reduce the computational time without loss of accuracy. Comparison between the experimental, empirical, and numerical results for inductance, using the direct and the indirect methods, indicates that the proposed formulas have high reliability. It is also shown that the self-inductance and mutual inductance could be calculated in the same way, provided that the radius of curvature is greater than 0.4 of the minor radius and that the definition of the geometric mean radius in the superconductivity conditions is used. Plotting contours for the magnetic flux density and the inductance show that the inductance formulas of the helical toroidal coil could be used as the basis for coil optimal design. Optimization target functions such as maximization of the ratio of stored magnetic energy with respect to the volume of the toroid or the conductor’s mass, the elimination or the balance of stress in certain coordinate directions, and the attenuation of leakage flux could be considered.

Index Terms—Behavioral study of magnetic flux density components, helical toroidal coil, inductance calculation, reactor plasmas, semitoroidal coordinate system, superconductor.

I. INTRODUCTION

RECENT research work on superconducting magnetic energy storage (SMES) systems, nuclear fusion reactors, and plasma reactors such as the Tokamak has suggested the use of advanced coil with a helical toroidal structure [1]–[4]. The main reason for this suggestion is the ability to implement the direct method is used as design parameters to satisfy special target functions. With respect to the fact that each ring of the coil generates both toroidal and poloidal magnetic fields simultaneously, the coil can be regarded as a combination of coils with toroidal and solenoid fields. Furthermore, the coil can be designed in a way to eliminate the magnetic force component in both the major and minor radius directions. These are called force- and stress-balanced coils, respectively. In addition, the coils that utilize the virial theorem to balance these two force components are called virial-limited coils [7]–[9]. In some applications, helical toroidal coils are used in a double-layer manner with two different winding directions (respectively with different or the same current directions in each layer) to reduce the poloidal leakage flux being compared to the toroidal leakage flux or vice versa. In this paper, the investigation is focused on the one-layer helical toroidal coil.

In general, any simulation program that simultaneously solves equations, the particle position, and its velocity can be called a particle-in-cell (PIC) simulation. The name PIC comes from the way of assigning macroquantities (like density, current density, and so on) to the simulation particles. Inside the plasma community, PIC codes are usually associated with solving the equation of motion of particles with the Newton–Lorenz’s force. PIC codes are usually classified depending on the dimensionality of the code and based on the set of Maxwell’s equations used. The codes solving an entire set of Maxwell’s equations are called electromagnetic codes, while electrostatic ones solve just the Poisson equation. Some advanced codes are able to switch between different dimensional and coordinate systems and use electrostatic or electromagnetic models. PIC

Fig. 1. Structure of a monolayer helical toroidal coil with five rings of nine turns.
simulation starts with an initialization and ends with the output of results. This part is similar to the input/output routines of any other numerical tool. Usually, the numerical methods based on the PIC simulation are obtained from the solution of partial differential–algebraic equations, for example, by the fourth-order Runge–Kutta method. Considering the number of particles which are on the order of $10^{10}$, the simulations based on PIC methods take a long time to solve the aforementioned equations. Usually, in order to resolve the time issue in the PIC methods, special computers may be employed. On the other hand, this paper used Biot–Savart equation and the mathematical equations of the current path in the conductor of the helical toroidal coil in order to obtain the magnetic flux density components. The numerical integrals resulting from the Biot–Savart equations are solved using the extended three-point Gaussian algorithms. This method has the least error among all numerical integration methods. In addition, the method used by the authors, contrary to the PIC method, does not need any special computers to solve the equations.

In Section II of this paper, the semitoroidal coordinate system, the longitudinal components of a ring element, and the ring radius of curvature are briefly discussed. In Sections III and IV, the formulas of the magnetic flux density components are presented and the behaviors of these components are simulated, respectively. In these sections, the variations of the ring radius of curvature with respect to the toroidal angle for the geometric parameters of the coil are also investigated. In Sections V and VI, the formulas for the self-inductance and mutual inductance of the helical toroidal coils using the direct and indirect methods are developed. Finally, in Section VII, the experimental inductance measurements are compared with the direct and indirect method simulation.

II. LONGITUDINAL COMPONENTS AND RADIUS OF CURVATURE OF A RING

The semitoroidal coordinate system is orthogonal, 3-D, and rotational (see Fig. 2). Longitudinal components of the ring element in this coordinate system can be defined by (1). Table I presents the dot products of the unit vectors for the Cartesian and the semitoroidal coordinates using the projection of these vectors on the planes $\varphi = \text{const}$ and $z = 0$

\[
\begin{align*}
dl &= \vec{a}_\theta dl_\theta + \vec{a}_\rho dl_\rho + \vec{a}_\varphi (R + a \cos \theta) d\varphi \\
&= \vec{a}_\rho dl_x + \vec{a}_\rho dl_y + \vec{a}_\rho dl_z.
\end{align*}
\]

Using Table I, the longitudinal components of the ring element in the Cartesian coordinate system can be defined by (2). As the ring's geometric loci is in the form of $\rho = \text{const}$, its differential $d\rho = 0$ can be replaced in (2). The relation between $\theta$ and $\varphi$ for a ring of $N$ turns can be expressed as (3) in which $\theta_0$ is the poloidal angle of the ring at plane $\varphi = 0$

\[
\begin{align*}
dl_x &= dl \cdot \vec{a}_x \\
&= -a \sin \theta \cos \varphi d\theta - \sin \varphi (R + a \cos \theta) d\varphi \\
&\quad + \cos \theta \cos \varphi d\rho \\
&= \frac{-a \sin \theta \sin \varphi d\theta + \cos \varphi (R + a \cos \theta) d\varphi + \sin \varphi \cos \theta d\rho}{\sqrt{f_4}} \\
&= \frac{dl_y}{\sqrt{f_4}}.
\end{align*}
\]

If the positional vector of each longitudinal element of the ring with respect to the origin of Cartesian coordinate system is defined as (4), then the tangent vector to the longitudinal element $T_\varphi(\varphi)$ and the radius of curvature of the ring can be defined as (5) and (6), respectively. In these equations, $f_1'$ and $f_2''$ are the first and second derivatives of function $f_1$ with respect to $\varphi$, respectively

\[
\begin{align*}
\vec{P}_\varphi(\varphi) &= f_1 \vec{a}_x + f_2 \vec{a}_y + f_3 \vec{a}_z \quad (4) \\
T_\varphi(\varphi) &= \vec{P}_\varphi(\varphi) / |\vec{P}_\varphi(\varphi)| \\
&= (f_1' \vec{a}_x + f_2' \vec{a}_y + f_3' \vec{a}_z) / \sqrt{f_4} \\
P_\varphi(\varphi) &= \left| \vec{P}_\varphi(\varphi) / T_\varphi(\varphi) \right| \\
&= f_2 / \left[ (f_1'' f_4 - f_1' f_5)^2 + (f_2'' f_4 - f_2' f_5)^2 \right]^{0.5} \\
&= a \cdot g(A, N)
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= (R + a \cos (N\varphi + \theta_0)) \cos \varphi \\
f_2 &= (R + a \cos (N\varphi + \theta_0)) \sin \varphi \\
f_3 &= a \sin (N\varphi + \theta_0) \\
f_4 &= f_1'^2 + f_2'^2 + f_3'^2 \\
f_5 &= f_1'' f_4 - f_1' f_5 \\
f_6 &= f_2'' f_4 - f_2' f_5.
\end{align*}
\]
III. Magnetic Flux Density Components

In this section, each infinite surface to which the normal vector \( \vec{a}_\varphi \) is perpendicular is called the toroidal plane, and each infinite surface (with conical shape) to which the normal vector \( \vec{a}_\theta \) is perpendicular is called poloidal cone. Therefore, each toroidal plane could be represented with the equation \( S_T : \varphi' = const \& \pi + const \), and each poloidal cone could be represented with the equation \( S_P : \theta' = const \& \pi + const \). It is noted that the apex of the cone for each poloidal cone is placed on the \( z \)-axis and that the surfaces of all poloidal cones include the geometric loci of origins for the semitoroidal coordinate systems. If the Cartesian coordinates of any point \( \beta \) on each toroidal or poloidal cone are \( x, y, \) and \( z \), and the Cartesian coordinates of any point \( \alpha \) on the ring with characteristics of \( \theta_\alpha, a, R, \) and \( N \) are presented by \( x_\alpha, y_\alpha, \) and \( z_\alpha \), then replacing (2), (3), and (7) in the Biot–Savart equation results in the Cartesian form of the magnetic flux density components of (8). In addition, it is possible to change these components into toroidal component of the magnetic flux density with (10) by defining point \( \beta \) on each toroidal plane using (9)

\[
\begin{align*}
x_\alpha &= (R + a \cos \theta) \cos \varphi \\
y_\alpha &= (R + a \cos \theta) \sin \varphi \\
z_\alpha &= a \sin \theta \quad (7)
\end{align*}
\]

\[
\begin{align*}
B_x &= (\mu_0 I/4\pi) \int_0^{2\pi} (f_7/f_6) d\varphi \\
B_y &= (\mu_0 I/4\pi) \int_0^{2\pi} (f_8/f_6) d\varphi \\
B_z &= (\mu_0 I/4\pi) \int_0^{2\pi} (f_9/f_6) d\varphi \quad (8)
\end{align*}
\]

where

\[
\begin{align*}
f_6 &= \left[ (x_\beta (R + a \cos(N \varphi + \theta_0)) \cos \varphi + (y_\beta (R + a \cos(N \varphi + \theta_0)) \sin \varphi + (z_\beta - a \sin(N \varphi + \theta_0))^2 \right]^{3/2} \\
f_7 &= \left[ (z_\beta - a \sin(N \varphi + \theta_0)) \cdot \left( (R + a \cos(N \varphi + \theta_0)) \cos \varphi - aN \sin(N \varphi + \theta_0) \sin \varphi \right) \right. \\
&\left. - (y_\beta -(R + a \cos(N \varphi + \theta_0)) \sin \varphi) \cdot (aN \cos(N \varphi + \theta_0)) \right] \\
f_8 &= \left[ (z_\beta - (R + a \cos(N \varphi + \theta_0)) \cos \varphi \cdot (aN \cos(N \varphi + \theta_0)) + (z_\beta - a \sin(N \varphi + \theta_0))(R + a \cos(N \varphi + \theta_0)) \sin \varphi \\
&+ aN \sin(N \varphi + \theta_0) \cos \varphi \right] \\
f_9 &= \left[ (y_\beta - (R + a \cos(N \varphi + \theta_0)) \sin \varphi \cdot (\sin \varphi (R + a \cos(N \varphi + \theta_0)) \\
&+ aN \sin(N \varphi + \theta_0) \cos \varphi - (R + a \cos(N \varphi + \theta_0)) \cos \varphi \\
&\cdot (\cos \varphi (R + a \cos(N \varphi + \theta_0)) - aN \sin(N \varphi + \theta_0) \sin \varphi) \right].
\end{align*}
\]

IV. Behavioral Study of the Magnetic Flux Density Components

In this section, the behavior of the magnetic flux density components for one ring or several rings is simulated using MATLAB. The numerical integrations in Sections III, V, and VI are performed using the extended three-point Gaussian algorithms [10]. This method has the least error among all numerical integration methods. In the sketching of the magnetic flux density components, it is assumed that

\[
\begin{align*}
\Delta \gamma &= \Delta \rho = 0.05 \text{ [m]} \\
\Delta \theta' &= \Delta \varphi' = \Delta \xi = 0.02 \text{ [rad]} \\
I &= 1 \text{ [kA]} \\
\rho_{\text{max}} &= R \\
\gamma_{\text{max}} &= 2R \\
\Delta \varphi &= \pi/150 \text{ [rad]}
\end{align*}
\]

Also in the calculation of the integral, the corresponding range is divided into \( n \approx 300 \) equal segments in order to reach the integration error of the derivative order \( 2n \) or \( 600 \) of the function under integration. It is obvious that, if the integration range is divided into more parts, the integration error will decrease further, but this comes at the cost of longer computational time. In Figs. 3–5, the contours for magnetic flux density, the toroidal, the poloidal, and the radial components are shown. It can be inferred that, as we approach the conductor’s geometric loci, the amplitude of the magnetic flux density becomes stronger. In these figures, the boundary of surface \( S_0 \), from which the magnetic flux density components only enter or exit, is marked with the number 0. Figs. 3(c) and (h), 4(c) and (f), and 5(e) and (f) show the behavior of the magnetic flux density components for \( \nu \) ring of \( N \) turns (total of \( N \nu \) turns) in comparison with a single \( N \nu \) turn ring. The comparison of the results for these two windings indicates that the magnetic flux
Fig. 3. Contours of toroidal component for magnetic flux density.
density components are different for these two case studies, and the reason for this difference should be sought in the winding types of these coils. Moreover, Fig. 3(a), (b), (d), and (e) shows that, by rotation of the ring, \( B \) maximum geometric loci is altering at \( \theta = N \varphi' + \theta_0 \). From Fig. 3(a), (c), and (g), it can be observed that the amplitude of \( B \) is increased with increasing \( v \) and \( N \) (increasing \( N \) is more efficient than \( v \)) and that the leakage flux percentage of \( B \) from the coil’s section will decrease. According to Fig. 3(a), (b), (d), and (e), the surface area \( S_0 \) on different toroidal planes is different. Furthermore, part of \( B \) is passing outside the coil’s section, and the maximum of \( B \) on different toroidal planes is altering for the same parameters \( a, R, \) and \( N \). Symmetry of the right half of \( B \) with the left half could be seen for the even values of \( N \), and the asymmetry of the right half of \( B \) with the left half is apparent for the odd or fractional values of \( N \) [see Fig. 3(a), (b), (d), and (e)]. In Fig. 3(f) and (g), it is shown that, in marginal conditions of \( N \approx 0 \) or \( N \gg 1 \), the coil behavior is approaching a pure solenoid or a toroidal coil, and the amplitude of \( B \) on each toroidal plane is decreased and increased, respectively. In Fig. 4, contours are plotted for \( B_0 \) on the poloidal cone of \( z = 0 \). Moreover, Fig. 4(a) and (b) shows that, by rotation of the ring, \( B_0 \), the maximum geometric loci is altering at \( \varphi = -\theta_0/N \). In addition, the maximum of \( B_0 \) is independent of \( \theta_0 \). From Fig. 4(a) and (c)–(f), it is inferred that the surface area \( S_0 \) is continuously changing and an increase in \( N \) increases the area of this surface. In other words, a percentage of \( B_0 \) is passing through the surface of the helical toroidal coil. In Fig. 4, the number of the rose leaves of the poloidal magnetic field is \( N \varphi \), and there are always \( N \varphi \) global maxima and minimums for \( B_0 \) on each poloidal cone. Fig. 4(a) and (c)–(f) shows that the amplitude of \( B_0 \) is increased and decreased with increasing \( v \) and \( N \), respectively. In other words, increasing \( v \) increases the density of \( B_0 \) amplitude maximum in the range \( a \leq \rho \leq R \). In Fig. 4(c) and (e), it is shown that, in the marginal conditions of \( N \gg 1 \) or \( N \approx 1 \), the coil behavior approaches the pure solenoid and the pure toroidal coils, respectively. Furthermore, the amplitude of \( B_0 \) is increased and decreased, respectively. It can be inferred from Fig. 4(g) that, in the marginal conditions of \( N = 0 \), the behavior of \( B_0 \) for the helical toroidal coil is similar to one ring, and the amplitude of \( B_0 \) is positive inside the ring and negative outside it.

In Fig. 5, the contours of the magnetic flux density of the radial component are shown. Fig. 5(a) and (b) shows that the \( B_{\rho} \) maximum on the different toroidal planes with the same values of \( a, R, \) and \( N \) is changing and occurs at \( \theta = N \varphi' + \theta_0 \). Fig. 5(a), (e), and (f) shows that the amplitude of \( B_{\rho} \) is increased with increasing \( v \) and \( N \) (increasing \( v \) is more efficient than \( N \)), and there is always \( N \varphi \) global maxima and minimums for \( B_{\rho} \) on each toroidal plane. It can be inferred from this figure that, due to its radial form, this component has no magnetic linkage with the ring. In Fig. 5(d) and (f), the amplitude of \( B_{\rho} \) in the marginal conditions of \( N = 0 \) and \( N \gg 1 \) on each toroidal plane is observed. In the same figures, the symmetry of this component with respect to poloidal cone \( z = 0 \) in the marginal condition of \( N = 0 \) is noticed.

In Fig. 6, the variations of the ratio radius of curvature of the ring to the coil minor radius with respect to the toroidal angle for parameters \( A \) and \( N \) are sketched. It can be inferred from Fig. 6 that the ratio of the radius of curvature to the minor radius increases with increasing the value of the aspect ratio in constant poloidal turns. In addition, the minimum of this value occurs at the aspect ratio of one for poloidal turn numbers of 0.4.

V. DIRECT CALCULATIONS OF SELF-INDUCTANCE AND MUTUAL INDUCTANCE

Equation (14) is introduced for the calculation of the mutual inductance between two rings with characteristics \( \theta_0, a_i, R_i, N_i, \) and \( \theta_0j, a_j, R_j, N_j \) based on the Neumann’s equation

\[
M_{ij} = M_{ij}^N = (\mu_0/4\pi) \int_0^{2\pi} \int_0^{2\pi} \left[ (f_{11}/f_{10}) \right] d\varphi_i d\varphi_j, \quad i \neq j
\]

(14)

where

\[
f_{10} = \left[ (a_i \sin(N_j \varphi_j + \theta_{0i}) - a_i \sin(N_i \varphi_i + \theta_{0i}))^2 \right.
\]

\[
+ ((R_j + a_j \cos(N_j \varphi_j + \theta_{0j})) \sin \varphi_j
\]

\[
- (R_i + a_i \cos(N_i \varphi_i + \theta_{0i})) \sin \varphi_i)^2
\]

\[
+ ((R_j + a_j \cos(N_j \varphi_j + \theta_{0j})) \cos \varphi_j
\]

\[
- (R_i + a_i \cos(N_i \varphi_i + \theta_{0i})) \cos \varphi_i)^2 \right]^{0.5}
\]

\[
f_{11} = [(a_i N_i \sin(N_i \varphi_i + \theta_{0i}) \cos \varphi_i
\]

\[
+ \sin \varphi_i (R_i + a_i \cos(N_i \varphi_i + \theta_{0i})))
\]

\[
\cdot (a_j N_j \sin(N_j \varphi_j + \theta_{0j}) \cos \varphi_j
\]

\[
+ \sin \varphi_j (R_j + a_j \cos(N_j \varphi_j + \theta_{0j})))
\]

\[
+ (-a_i N_i \sin(N_i \varphi_i + \theta_{0i}) \sin \varphi_i
\]

\[
+ \cos \varphi_i (R_i + a_i \cos(N_i \varphi_i + \theta_{0i})))
\]

\[
\cdot (-a_j N_j \sin(N_j \varphi_j + \theta_{0j}) \sin \varphi_j
\]

\[
+ \cos \varphi_j (R_j + a_j \cos(N_j \varphi_j + \theta_{0j})))
\]

\[
+ a_i N_i \cos(N_i \varphi_i + \theta_{0i}) \cdot a_j N_j \cos(N_j \varphi_j + \theta_{0j})).
\]

According to classical electrodynamics, if the radius of the curvature is larger than the dimensions of the transverse section of the conductor (i.e., the diameter of the conductor’s cross section is smaller than 0.4 of the minor radius of the helical toroidal coil), the equation of the mutual inductance between the two rings can be used to calculate the self-inductance of the ring. In this condition, the minimum distance between the two corresponding points in each ring is assumed to be equal to the geometrical mean radius of the conductor’s cross section. The geometrical mean radius of the conductor’s cross section is, in fact, the efficient radius of the conductor that the magnetic
Fig. 4. Contours of the poloidal component of the magnetic flux density.

In nonsuperconductivity conditions, the geometric mean radius of the conductor’s cross section with radius $r$ is defined as $r_m = r e^{-0.25}$. In superconductivity conditions, the geometric mean radius is assumed to be $r_m = r$, because the magnetic field cannot influx the conductor’s cross section. Therefore, (15) with its assumptions can be used to calculate the self-inductance of the $i$th ring with characteristics $\theta_{0i}, a, R$, and $N$

$$L_{ii} = M_{ii}, \quad i = 1, \ldots, v$$

$$R_i = R_j = R, \quad a_i = a_j = a, \quad N_i = N_j = N$$

$$\theta_{0i} = \theta_{0j} \text{ [rad]}, \quad \theta_{0j} = \theta_{0i} + (r_m/a) \text{ [rad]}.$$

(15)
VI. CALCULATIONS OF SELF-INDUCTANCE AND MUTUAL INDUCTANCE USING INDIRECT METHOD

In this section, the mutual and self-inductances of the helical toroidal coil are proposed by the indirect method or the magnetic flux density components. The self-inductance of one ring or the mutual inductance between two rings is proportional to the surface integral of the magnetic flux density components of one or two rings. The radial component of the magnetic flux density is perpendicular to the ring geometric loci and does not have magnetic link to one ring or two rings. Consequently, this does not affect the calculation of the mentioned inductances. In other words, the only effective components in the calculation of the self-inductance and mutual inductance are the magnetic flux density of toroidal and poloidal components.

A. Calculation of Mutual Inductance via Indirect Method

In this section, the mutual inductances of the \(i\)th and the \(j\)th rings with characteristics of \(\theta_{0i}, a_i, R_i, N_i\) and \(\theta_{0j}, a_j, R_j, N_j\), with incorporation of some engineering assumptions, are calculated. The pure flux of field without divergence on each closed surface is zero. The magnetic field entering into each toroidal or poloidal cone could be calculated as of surface integration of the absolute values of \(B_\theta\) and \(B_\phi\) on these planes. Consequently, it is possible to estimate the values of these
The toroidal and the poloidal mutual inductances are the result of equations:

\[ \psi = \text{functions of integrals with the linking flux of the toroidal angle.} \]

Fig. 6. Variations of radius of curvature of the ring with respect to the toroidal angle.

integrals with the linking flux of the \( i \)th ring using the following equations:

\[
\psi_T(\varphi') = 0.5 \int_{S_T} |B_{\varphi'}| ds_{\varphi'} = 0.5 \int_0^{2\pi} \int_0^\infty |B_{\varphi'}| \rho d\varphi' d\rho
\]

\[
\psi_P(\theta') = 0.5 \int_{S_P} |B_\theta| ds
= 0.5 \left( \int_0^{2\pi} - R_i / \cos \theta' \right) \int_0^\infty |B_\theta| (R_i + \rho \cos \theta') d\rho d\varphi'
+ \int_0^{2\pi} \int_0^\infty |B_\theta| (R_i + \rho \cos \theta') d\rho d\varphi'.
\]  

(16)

(17)

On the one hand, these equations indicate that \( \psi_T \) and \( \psi_P \) are functions of \( \varphi' \) and \( \theta' \), respectively. On the other hand, the toroidal and the poloidal mutual inductances are the result of \( \psi_T \) and \( \psi_P \) generated by the \( i \)th ring linking with the cross section of \( S_{TM} \) and \( S_{PM} \). These surfaces could be stated as (18) and (19).

It is important to mention that the linking magnetic fluxes with the cross section of \( S_{TM} \) and \( S_{PM} \) link \( N_i \) and one ring and are defined as (20) and (21), respectively. Based on the definition of \( S_{TM} \), it is necessary to apply the definition of point \( \beta \) based on (11) for replacement of \( B_\varphi \) in (16) and (20)

\[
S_{TM} : 0 \leq \rho \leq a_j, \quad 0 \leq \theta' \leq 2\pi, \quad \varphi' = \text{const}, \pi + \text{const}
\]

\[
S_{PM} : \begin{cases}
0 \leq \varphi' \leq 2\pi & \text{or} \quad a_j \leq \rho \leq \infty \\
-\pi/2 \leq (\theta' = \text{const}, \pi + \text{const}) \leq \pi/2 & 0 \leq \varphi' \leq 2\pi
\end{cases}
\]

\[
\psi_{TM}(\varphi') = N_j \int_{S_{TM}} |B_\varphi| ds = N_j \int_0^{2\pi} \int_0^\infty |B_\varphi| \rho d\varphi' d\rho
\]

\[
\psi_{PM}(\theta') = \int_{S_{PM}} |B_\theta| ds
= \int_0^{2\pi} - R_i / \cos \theta' \left( \int_0^\infty |B_\theta| (R_i + \rho \cos \theta') d\rho d\varphi' \right).
\]

(20)

(21)

Toroidal angle \( \varphi' = \varphi_{op} \) and poloidal angle \( \theta' = \theta_{op} \) could be achieved by the derivation of (20) and (21), which minimizes \( \psi_{TM} \) and \( \psi_{PM} \). Then, the toroidal and poloidal mutual inductions could be defined as (22) and (23). As it was mentioned before, \( \psi_T \) and \( \psi_P \) must be particularly independent of the variations of \( \varphi' \) and \( \theta' \). In other words, the variations of \( \psi_T \) and \( \psi_P \) must be negligible in comparison with the variations of \( \varphi' \) and \( \theta' \). Therefore, in order to be able to simplify the equations, the assumptions of \( \varphi_{op} = 0 \) and \( \theta_{op} = \pi \) are important. By implementing these assumptions, the toroidal and the poloidal mutual impedances can be defined according to (24) and (25)

\[
M_{ijT} = \psi_{TM}(\varphi' = \varphi_{op}) / I
\]

\[
M_{ijP} = \psi_{PM}(\theta' = \theta_{op}) / I
\]

\[
M_T = \psi_{TM}(\varphi' = \varphi_{op} = 0) / I
\]

\[
M_P = \psi_{PM}(\theta' = \theta_{op} = \pi) / I.
\]

(22)

(23)

(24)

(25)

B. Calculation of Self-Inductance via Indirect Method

In this section, the self-inductance of the ring with characteristics of \( \theta_{0i}, a_i, R_i, \text{and} N_i \) is calculated with the following assumptions.

1) \( \psi_T \) and \( \psi_P \) are averaged.
2) Leakage fluxes are negligible.
3) \( \psi_T \) and \( \psi_P \) are independent of the variations of \( \varphi' \) and \( \theta' \), or assuming \( \varphi' = \varphi_{mean} = 0 \) and \( \theta_{op} = \theta_{mean} = \pi \), the calculations are simplified.
The parameters of the helical toroidal coil with geometrical to-center distance of 10 cm are reported as $0.24879 \, \mu \text{H}$. The empirical results for the mutual inductance characteristics of the ring is simulated using MATLAB. In addition, with respect to the fact that the currents for $M_{ijP}$ and $M_{ijT}$ are equal, it can be inferred that the mutual inductance of the rings $i$ and $j$, $M_{ij}$, could be obtained from the summation of $M_{ijT}$ and $M_{ijP}$.

Electrical circuit analysis shows that the inductance matrix of a monolayer helical toroidal coil could be obtained using filament method as in (34), and the inductance of one layer, assuming $L_{ii} = M_{ii}$, could be obtained by multiplying $\nu^2$ by $L_{ii}$ or $M_{ij}$.

$$L_{	ext{coll}} = \begin{bmatrix}
L_{11} & M_{12} & \cdots & M_{1\nu} \\
M_{21} & L_{22} & \cdots & M_{2\nu} \\
\vdots & \vdots & \ddots & \vdots \\
M_{1\nu} & M_{2\nu} & \cdots & L_{\nu\nu}
\end{bmatrix} \quad (34)$$

$$L_{	ext{coll}} = \nu^2 (L_{ii} = M_{ij}). \quad (35)$$

VII. COMPARISON OF EXPERIMENTAL, EMPIRICAL, AND NUMERICAL RESULTS

In this section, the behavioral study of the inductance characteristics of the ring is simulated using MATLAB. In addition, the empirical results and experimental results of the induction measurements are compared with their corresponding numerical values. The empirical results for the mutual inductance between two flat rings of radius 20 and 25 cm with center-to-center distance of 10 cm are reported as 0.24879 $\mu \text{H}$ [11]. The parameters of the helical toroidal coil with geometrical calculations could be obtained as (36) to adapt this problem with equations mentioned in Section V.

$$N = N_1 = N_2 = 0 \, \text{[turns]}$$
$$R = R_1 = R_2 = 22.5 \, \text{[cm]}$$
$$a = a_1 = a_2 = \sqrt{125} / 2 \, \text{[cm]}$$
$$\theta_{01} = -\tan^{-1}(2) \, \text{[rad]}$$
$$\theta_{02} = \pi - \tan^{-1}(2) \, \text{[rad]}.$$

$$L = \mu_0 \tau \left[ Ln(8\pi / r) - 1.75 \right] \, \text{[H]} \quad (37)$$
$$R + a = 200 \, \text{[cm]} \quad r = 1 \, \text{[cm]}$$
$$\theta_{01} = 0 \, \text{[rad]} \quad \theta_{02} = re^{-0.25/a} \, \text{[rad]}.$$
The experimental result of helical toroidal coil inductance with geometric characteristics shown in Table II is 17 mH [6]. It can be observed that the experimental results are in good agreement with those obtained from (35), with an error of less than 2.3%. The error can be due to the measurement, value of $n$, assumptions made in Section VI, and $M_{ij} = M_{ji}$. The convergence diagram, the simulation time, and the error percentage versus $n$ for numerical and experimental results are shown in Figs. 11 and 12, respectively. It can be inferred from Figs. 7, 9, and 11 that the optimal value of $n$ in the indirect method is nearly half of that in the direct method. As a result, the simulation time in the indirect method is nearly half of that in the direct method. Therefore, one can conclude that the equations presented for the inductance calculations, accepting calculation error of 2.3% (acceptable engineering error), with the assumptions in the indirect method being compared to that in the direct method, have higher reliability with less simulation time.

VIII. CONCLUSION

Helical toroidal coils are superior to other coils and are extensively used in the SMES systems, nuclear fusion reactors, and plasma research work. Considering the complexity of the coils and the fact that not much investigation has been carried out in this field, this area of research is still open for much more academic work to come. On the other hand, the calculation of inductance for these types of coils can be an index to determine the behavior of the transient state, determination of
the electrical equivalent circuit, and estimation of values for electrical elements of the coil equivalent circuit [14]–[24].

In this paper, the inductance of the helical toroidal coil is calculated in the direct and indirect methods, and the inductance characteristics and the magnetic flux density are simulated in MATLAB. Comparison of the experimental, empirical, and numerical results shows that the equations for inductance calculations have great reliability and that dividing the inductance for one ring into two toroidal and poloidal components with incorporation of some engineering assumptions simplifies the equations and decreases the computational time without the loss of accuracy.

REFERENCES


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